From MHV amplitudes to the CHY formulation

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Based on works with Freddy Cachazo & Ellis Yuan

MHV @ 30, Fermilab

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MHV amplitudes & Parke-Taylor formula

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A miracle for n-gluon scattering [Parke,Taylor '86; Mangno, Parke, Xu '87]

$$M_n(1^+, \ldots, i^-, \ldots, j^-, \ldots, n^+) = \frac{\langle i\, j \rangle^4}{\langle 1\, 2\rangle \langle 2\, 3\rangle \cdots \langle n{-}1\; n\rangle \langle n\, 1\rangle}\,.$$

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Twistor-string formulas for $\mathcal{N}=8$ supergravity (PT replaced by Hodges determinants/BGK formula) [Cachazo, Skinner '12...]

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Deep connections to string theory; ambi-twistor strings or chiral, infinite-tension limit [Mason, Skinner '13; Berkovits '13; Siegel '15...]

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- U(1) decoupling identity (and Kleiss-Kuijf relations)

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- Bern-Carrasco-Johansson (BCJ) partial-amp relations (?):

$$s_{1\,2}PT(2,1,3,\dots,n)+(s_{1\,2}+s_{1\,3})PT(2,3,1,\dots,n) \\ +\dots+(s_{1\,2}+\dots+s_{1\,n-1})PT(2,\dots,n-1,n)=0\,,$$

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  if and only if $\sum_{b=2}^{n}\frac{s_{1\,b}(*\,b)}{(1\,b)}=0$! Similarly for $a=2,\ldots,n$.
- Satisfied on support of twistor-string constraints [Cachazo '12]. BCJ universal: consider these equations in general!

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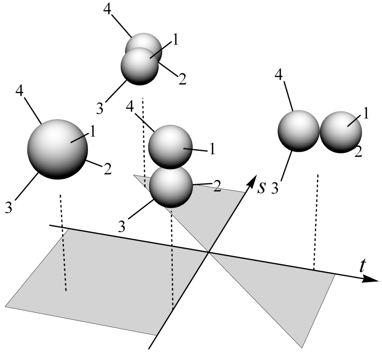
$\Rightarrow$ massless tree amps from solutions of the equations on $\mathcal{M}_{0,n}$

Four points



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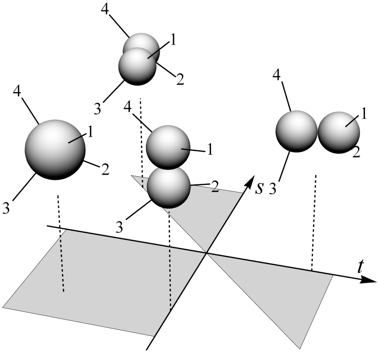

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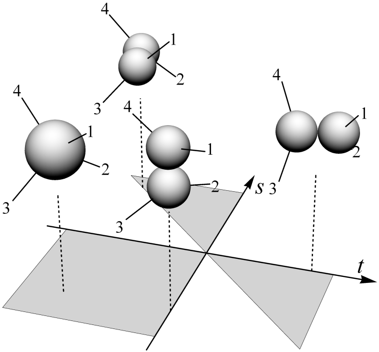

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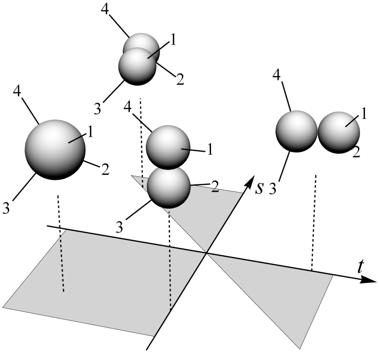

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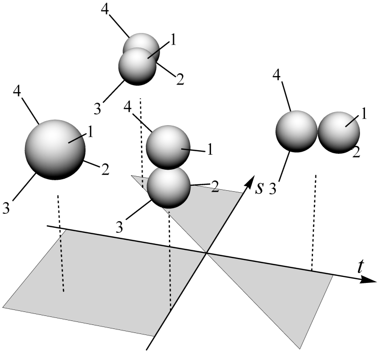

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Higher points: multi-factorizations vs. higher-dim singularities

CHY formulation

Tree amps = contour integral in $\mathcal{M}_{n,0}$ = sum over solutions

$$\boxed{M_n = \int \underbrace{\frac{d^n\sigma}{\text{vol SL}(2,\mathbb{C})} \prod_a{}' \delta(E_a)}_{d\mu_n} \mathcal{I}(\{k,\epsilon,\sigma\}) = \sum_{\{\sigma\}\in\text{solns.}} \frac{\mathcal{I}(\{k,\epsilon,\sigma\})}{J(\{\sigma\})}}$$

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- "proof": factorization + soft limits [also see Goddard, Dolan '13]

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$$\boxed{m[\pi|\rho] := \int \frac{d^n \sigma_a}{\text{vol.}} \prod_a{}' \delta\Big(\sum_{b \neq a} \frac{k_a \cdot k_b}{\sigma_a - \sigma_b}\Big)\, \mathrm{PT}[\pi]\, \mathrm{PT}[\rho]}$$

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What does the formula compute?

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$m[\pi|\rho]$ computes the sum of trivalent scalar diagrams (massless propagators) that are consistent with both π, ρ orderings

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$$m[1234|1243] = \frac{1}{s_{12}}, \, m[1234|1324] = \frac{1}{s_{14}}, \, m[1234|1234] = \frac{1}{s_{12}} + \frac{1}{s_{14}}$$

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Trivalent diagrams from CHY

$m[\pi|\rho]$ computes the sum of trivalent scalar diagrams (massless propagators) that are consistent with both π, ρ orderings

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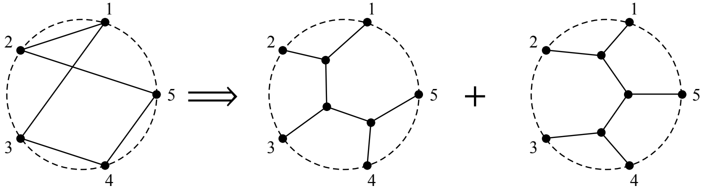

Simplest CHY formula: ϕ^3 theory

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CHY formula for bi-adjoint ϕ^3 amplitudes: gives sum of all $m[\pi|\rho]$'s with flavor factors (note permutation invariance)

$$M_n^{\phi^3} = \int d\mu_n \, \mathcal{C} \, \mathcal{C}' = \sum_{\pi,\rho} \mathrm{Tr}(T^{I_{\pi(1)}} \cdots T^{I_{\pi(n)}}) \, \mathrm{Tr}(T^{I_{\rho(1)}} \cdots T^{I_{\rho(n)}}) \, m[\pi|\rho]$$

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$$A_{a,b} := \begin{cases} \frac{k_a \cdot k_b}{\sigma_{a,b}} & a \neq b \\ 0 & a = b \end{cases}, \quad B_{a,b} := \begin{cases} \frac{\epsilon_a \cdot \epsilon_b}{\sigma_{a,b}} & a \neq b \\ 0 & a = b \end{cases},$$

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The other copy is the Parke-Taylor factor, or $\mathcal{C}$ for colors:

$$M_n^{\mathrm{YM}}[\pi] = \int d\mu_n \, \mathrm{PT}[\pi] \, \mathrm{Pf}' \, \Psi \Rightarrow \mathcal{M}_n^{\mathrm{YM}} = \int d\mu_n \, \mathcal{C} \, \mathrm{Pf}'\Psi$$

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Complete S-matrix for any number of gluons in any dimension

The origin of $\mathrm{Pf}'\Psi$: by scattering equations, it is exactly given by open-string correlators in the field-theory limit

$$\mathrm{Pf}'\Psi \sim \langle V^{(0)}(\sigma_1)\ldots V^{(-1)}(\sigma_i)\ldots V^{(-1)}(\sigma_j)\ldots V^{(0)}(\sigma_n)\rangle$$

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\end{array}\right)$$

Substituting $\epsilon_1 \to k_1$ $\mathrm{Pf}'\Psi = 0$ for each solution of scattering equations $\Longrightarrow$ gauge invariance manifest from CHY formula!

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$$\boxed{M_n^{h+B+\phi} = \int d\mu_n \, \mathrm{Pf}'\Psi(\epsilon) \, \mathrm{Pf}'\Psi(\epsilon') \longrightarrow M_n^{\mathrm{GR}} = \int d\mu_n \, \det{}' \, \Psi(\epsilon)}$$

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The third way of seeing "GR = YM$^2/\phi^3$" after [KLT '86, BCJ' 08].

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$\mathrm{Pf}'\Psi(\epsilon) \times \mathrm{Pf}'\Psi(\epsilon')$ correspond to closed-string correlator by using scattering equations: closed-string = open-string [2]

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Amplitude as a sum of $(n-3)!$ “virtual amplitudes”:

$$M_n = \sum_{I=1}^{(n-3)!} \frac{\mathcal{I}_n^{(I)}(\{k,\epsilon\})}{J_n^{(I)}(\{s_{a\,b}\})} := \sum_{I=1}^{(n-3)!} V_n^{(I)}(\{k,\epsilon\})$$

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Alternatively, $\mathrm{Pf}'\Psi = \sum_\alpha N_\alpha PT[\alpha] \;\rightarrow\;$ local, trivalent-graph expansion with BCJ numerators (but gauge variant).

More theories in CHY

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We can generate new formulas from old ones, e.g. compactify $R^{d+m} \to R^d$ with $K = (k^{(d)} \mid 0)$, $\mathcal{E} = (\epsilon^{(d)} \mid 0)$ or $(0 \mid e^{(m)})$:

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A corollary of the latter is a intriguing formula for ϕ^4 theory.

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The strangest is a special Galileon theory (a scalar theory with many derivatives) [Cheung et al '14,...], $M_n^{\text{sGal}} = \int d\mu_n (\mathrm{Pf}'A)^4$.

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For NLSM, scalar DBI, sGal $M_n \sim (\mathrm{Pf}'A)^2, (\mathrm{Pf}'A)^3, (\mathrm{Pf}'A)^4$, with soft emission $p^\mu \sim \tau \to 0$, $M_n \sim \tau^1, \tau^2, \tau^3 \to 0$.

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For NLSM, scalar DBI, sGal $M_n \sim (\mathrm{Pf}'A)^2, (\mathrm{Pf}'A)^3, (\mathrm{Pf}'A)^4$, with soft emission $p^\mu \sim \tau \to 0$, $M_n \sim \tau^1, \tau^2, \tau^3 \to 0$.

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Hidden simplicity of these special EFT's (∞ vertices): enhanced soft behavior play the role of gauge/diff invariance.

More theories in CHY

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Another operation: introducing non-abelian interactions

EM $\to$ Einstein-Yang-Mills, YMs $\to$ YMs $+ \phi^3$

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"Parks-Taylor-like" factors are ubiquitous: one factor for each trace of gluons e.g. single-trace and (pure-gluon) double-trace

$$M_n^{(1,\ldots,m)}(g;h) = \int d\mu_n\, PT(1,\ldots,m)\, \mathrm{Pf}[\Psi]_h\, \mathrm{Pf}'\Psi(\tilde{\epsilon})\,,$$
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Repeating the operation: compact formula for all multi-trace gluon-graviton amps in EYM (or YMs $+\ \phi^3$).

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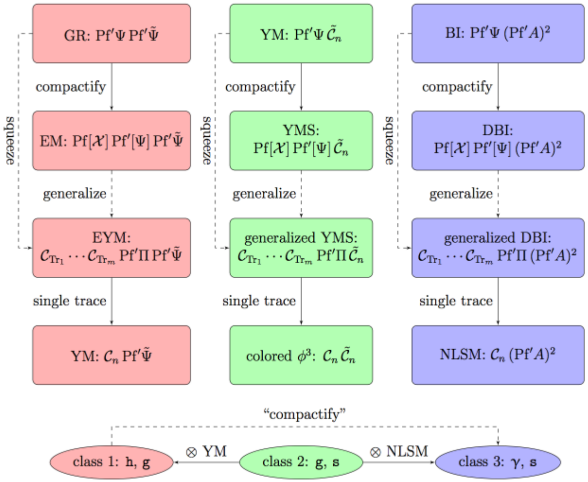
New ambitwistor-string models [Geyer et al '15]. String origin?

A landscape of massless theories



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Rewrite CHY formula (with two half-integrands), in terms of two $(n-3)!$-dim vectors $\mathbf{L}$, $\mathbf{R}$ and a diagonal matrix $\mathbf{J}$

$$M_n = \int d\mu_n \, L \, R \; = \; \sum_{I=1}^{(n-3)!} \frac{L_I \; R_I}{J_I} = \mathbf{L} \cdot \mathbf{J}^{-1} \cdot \mathbf{R} \,,$$

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Remarkable role of PT's: provide a new $(n-3)!$-dim basis

$$(n-3)!-\text{dim matrix } \mathbf{E} : \quad E_I^{\alpha} = PT[\alpha]_I \, , \quad \text{for } \alpha \in S_{n-3} \, .$$

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Applying it to partial amps $\rightarrow$ BCJ as basis expansion:
$M_n[\pi] = \sum_{\alpha,\beta \in S_{n-3}} m[\pi|\alpha]\; m^{-1}[\alpha|\beta]\; M_n[\beta]$, for $\pi \in S_n$.

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$q \rightarrow 0$: one-loop scattering eqs on a sphere [Geyer et al '15]

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Imposing $\delta(\mathcal{E})$'s gives formula for one-loop amplitudes

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New rep of loop integrands: a rational function with no ambiguities (treat all propagators equally) [c.f. Baadsgaard et al '15]

Loops from trees

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One-loop amp as forward limit of tree amp in higher dim:

$$M_n^{1-\text{loop}} \sim \int \frac{d^D \ell}{\ell^2} \sum_{I_+ = I_-, \epsilon_+ = (\epsilon_-)^*} M_{n+2}^{\text{tree}}(\ \{(k_i; 0)\},\ \pm(\ell, |\ell|)\ ),$$

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Color-sum gives one-loop color structures → one-loop PT's

$$\mathrm{PT}_n^{(1)}[1,2,\ldots,n] := \sum_{i=1}^{n} \mathrm{PT}_{n+2}[1,\ldots,i,+,-,i{+}1,\ldots,n]\,.$$

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One-loop "Pfaffians" from forward-limit of tree ones, e.g.

$$\mathrm{Pf}_{\mathbf{s}}^{(1)} = \frac{1}{\sigma_{+,-}^2}\mathrm{Pf}\Psi_n(\ell), \quad \mathrm{Pf}_{\mathbf{g}}^{(1)} = \sum_{\epsilon_+=(\epsilon_-)^*} \mathrm{Pf}'\Psi_{n+2}(\ell), \quad \mathrm{Pf}_{\mathbf{f}}^{(1)} = \ldots$$

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Formulas for ϕ^3, Yang-Mills and gravity at one loop

$$\mathcal{I}_n^{\phi^3} = (\mathrm{PT}_n^{(1)})^2, \quad \mathcal{I}_n^{\mathrm{YM}} = \mathrm{PT}_n^{(1)}\,\mathrm{Pf}_{\mathbf{g}}^{(1)}, \quad \mathcal{I}_n^{\mathrm{GR}} = (\mathrm{Pf}_{\mathbf{g}}^{(1)})^2 - c_d(\mathrm{Pf}_{\mathbf{f}}^{(1)})^2\,,$$

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Gauge invariance, soft theorems, unitarity cuts, SUSY ...
natural one-loop KLT and BCJ relations at integrand level: e.g.

$$\mathrm{SUGRA} = \sum_{\alpha,\beta=1}^{(n-1)!-2(n-2)!} \mathrm{SYM}[\alpha]\,(\phi_3)^{-1}[\alpha|\beta]\,\mathrm{SYM}[\alpha]\,.$$

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Most importantly: what is the origin of this formulation? Relations/applications to string theory & quantum gravity?

Thank you for your attention!