General Solution of the Scattering Equations

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(work with Peter Goddard, IAS Princeton)

1511.09441 [hep-th], *General Solution of the Scattering Equations*

1402.7374 [hep-th], *The Polynomial Form of the Scattering Equations*

1311.5200 [hep-th], *Proof of the Formula of Cachazo, He and Yuan for Yang-Mills Tree Amplitudes in Arbitrary Dimension*

Freddy Cachazo, Song He, and Ellis Yuan (CHY)

1309.0885 [hep-th], *Scattering of Massless Particles: Scalars, Gluons and Gravitons*

1307.2199 [hep-th], *Scattering of Massless Particles in Arbitrary Dimensions*

1306.6575 [hep-th], *Scattering Equations and KLT Orthogonality*

Carlos Cardona and Chrysostomos Kalousios

1511.05915 [hep-th] *Elimination and Recursions in the Scattering Equations*

Outline

- Tree amplitudes for Yang-Mills and massless ϕ^3 theory from the Scattering Equations in any dimension

- Möbius invariance

- Polynomial form of the Scattering Equations

- The General Solution and Elimination Theory

Tree Amplitudes

$$\mathcal{A}(k_1, k_2, \ldots, k_N) = \oint_{\mathcal{O}} \Psi_N(z, k, \epsilon) \prod_{a \in A}{}' \frac{1}{f_a(z,k)} \prod_{a \in A} \frac{dz_a}{(z_a - z_{a+1})^2} / d\omega,$$

$\mathcal{O}$ encircles the zeros of $f_a(z, k)$,

$$f_a(z,k) \equiv \sum_{\substack{b \in A \\ b \neq a}} \frac{k_a \cdot k_b}{z_a - z_b} = 0 \qquad \text{The Scattering Equations}$$

(*Cachazo*, *He*, *Yuan* 2013) … (*Fairlie*, *Roberts* 1972)

$$k_a^2 = 0, \quad \sum_{a \in A} k_a^\mu = 0, \quad A = \{1, 2, \ldots N.\}$$

Motivated by twistor string theory, DG proved $\mathcal{A}(k_1, k_2, \ldots k_n)$ are ϕ^3 and Yang-Mills gluon field theory tree amplitudes, as conjectured by CHY.

Scattering Equations $\quad f_a(z,k) = 0, \quad k_a^2 = 0 \quad z_a \to \frac{\alpha z_a + \beta}{\gamma z_a + \delta}$,

$$U(z,k) \equiv \prod_{a<b} (z_a - z_b)^{-k_a \cdot k_b} \quad \text{is Möbius invariant,}$$

$$\frac{\partial U}{\partial z_a} = -f_a U, \quad f_a(z,k) = \sum_{\substack{b \in A \\ b \neq a}} \frac{k_a \cdot k_b}{z_a - z_b},$$

implying $f_a(z) \to f_a(z) \dfrac{(\gamma z_a + \delta)^2}{(\alpha\delta - \beta\gamma)}$.

The infinitesimal transformations $\delta z_a = \epsilon_1 + \epsilon_2 z_a + \epsilon_3 z_a^2$,

$U(z + \delta z) \sim U(z) + \dfrac{\partial U}{\partial z_a} \delta z_a$, so the f_a satisfy the three relations

$$\sum_{a \in A} f_a = 0, \quad \sum_{a \in A} z_a f_a = 0, \quad \sum_{a \in A} z_a^2 f_a = 0.$$

There are $N-3$ independent Scattering Equations $f_a = 0$.
Fixing $z_1 = \infty, z_2 = 1, z_N = 0$, there are $N-3$ variables, and generally $(N-3)!$ solutions $z_a(k)$.

Total Amplitudes

$\Psi_N = \prod_{a\in A}(z_a - z_{a+1}) \times$ Pfaffian for Yang-Mills
For example, $N = 4$,

$$A^{abcd}(k_1, k_2, k_3, k_4) = g^2\Big(f_{abe}f_{ecd}\,\frac{n_s}{s} + f_{bce}f_{ead}\,\frac{n_t}{t} + f_{cae}f_{ebd}\,\frac{n_u}{u}\Big)$$
$$= g^2\Big(\big(tr(T_aT_bT_cT_d) + tr(T_dT_cT_bT_a)\big)\,A(1234)$$
$$+ \big(tr(T_aT_cT_dT_b) + tr(T_bT_dT_cT_a)\big)\,A(1342)$$
$$+ \big(tr(T_aT_dT_bT_c) + tr(T_cT_bT_dT_a)\big)\,A(1423)\Big),$$

$$n_s = \big(\epsilon_1\cdot\epsilon_2(k_1 - k_2)_\alpha + 2\epsilon_1\cdot k_2\epsilon_{2\alpha} - 2\epsilon_2\cdot k_1\epsilon_{1\alpha}\big)$$
$$\times\big(\epsilon_3\cdot\epsilon_4(k_3 - k_4)^\alpha + 2\epsilon_3\cdot k_4\epsilon_4^\alpha - 2\epsilon_4\cdot k_3\epsilon_3^\alpha\big)$$
$$+ \big(\epsilon_1\cdot\epsilon_3\epsilon_2\cdot\epsilon_4 - \epsilon_1\cdot\epsilon_4\epsilon_2\cdot\epsilon_3\big)\,s,$$

$$A(1234) = \frac{n_s}{s} + \frac{n_t}{t}. \qquad s = (k_1 + k_2)^2, t = (k_2 + k_3)^2, u = (k_1 + k_3)^2$$

$$\mathcal{A}(k_1, k_2, k_3, k_4) = A(1234).$$

A Single Scalar Field, Massless ϕ^3

A single massless scalar field, $\Psi_N = 1$.

$$\mathcal{A}^{\phi}(k_1, k_2, \ldots, k_N) = \oint_{\mathcal{O}} \prod_{a \in A}{}' \frac{1}{f_a(z,k)} \prod_{a \in A} \frac{dz_a}{(z_a - z_{a+1})^2} / d\omega$$

$$\mathcal{A}^{\phi}(k_1, k_2, k_3, k_4) = \frac{1}{s} + \frac{1}{t},$$

$$A^{\text{total}} = \mathcal{A}^{\phi}(k_1, k_2, k_3, k_4) + \mathcal{A}^{\phi}(k_1, k_3, k_2, k_4) + \mathcal{A}^{\phi}(k_1, k_4, k_2, k_3)$$

$$= 2\left(\frac{1}{s} + \frac{1}{t} + \frac{1}{u}\right)$$

Rewriting the Scattering Equations as Polynomial Equations whose degrees are as small as possible:

For a subset $U \subset A$,

$$k_U \equiv \sum_{a \in U} k_a, \qquad z_U \equiv \prod_{b \in U} z_b,$$

then the Scattering Equations

$$\sum_{\substack{b \in A \\ b \neq a}} \frac{k_a \cdot k_b}{z_a - z_b} = 0$$

are equivalent to the homogeneous polynomial equations

$$\sum_{\substack{U \subset A \\ |U| = m}} k_U^2 z_U = 0, \quad 2 \leq m \leq N-2,$$

where the sum is over all $\frac{N!}{m!(N-m)!}$ subsets $U \subset A$ with m elements.

Proof of the Polynomial Form of the Scattering Equations

$$p^\mu(z) \equiv \sum_{a\in A} \frac{k_a^\mu}{z - z_a}, \quad \sum_a k_a^\mu = 0, \quad k_a^2 = 0,$$

$$p^2(z) = \sum_{a,b} \frac{k_a \cdot k_b}{(z - z_a)(z - z_b)} = \frac{1}{2} \sum_a \frac{1}{z - z_a} \sum_{b\neq a} \frac{k_a \cdot k_b}{(z_a - z_b)} = 0$$

$$\begin{aligned} 2p^2(z) \prod_{c\in A} (z - z_c) &= \sum_{a,b\in A} k_a \cdot k_b \prod_{\substack{c\in A \\ c\neq a,b}} (z - z_c) \\ &= \sum_{m=0}^{N-2} z^{N-m-2} \sum_{\substack{U\subset A \\ |U|=m}} z_U \sum_{\substack{S\subset \overline{U} \\ |S|=2}} k_S^2 = 0 \end{aligned}$$

where $\overline{U} = \{b \in A : b \notin U\}$. Using $\sum_{\substack{S\subset \overline{U} \\ |S|=2}} k_S^2 = k_{\overline{U}}^2 = k_U^2$, then

$\widetilde{h}_m \equiv \sum_{\substack{U\subset A \\ |U|=m}} k_U^2 z_U = 0.$

$\widetilde{h}_m = 0$ are the Unique Möbius Invariant Polynomial Equations

L_{-1} denotes the generator of translations,

$$L_{-1} = -\sum_{a\in A} \frac{\partial}{\partial z_a}, \qquad L_{-1}\,\tilde{h}_m = -(N-m-1)\tilde{h}_{m-1}\,,$$

L_1, special conformal transformations

$$L_1 = -\sum_{a\in A} z_a^2 \frac{\partial}{\partial z_a} + \Sigma_1^A, \qquad L_1\,\tilde{h}_m = (m-1)\tilde{h}_{m+1}, \qquad \Sigma_1^A \equiv \sum_{a\in A} z_a.$$

L_0, scale transformations

$$L_0 = -\sum_{a\in A} z_a \frac{\partial}{\partial z_a} + \frac{N}{2}, \qquad L_0\,\tilde{h}_m = (\tfrac{1}{2}N-m)\tilde{h}_m\,,$$

so that $\quad [L_1, L_{-1}] = 2L_0, \qquad [L_0, L_{\pm 1}] = \mp L_{\pm 1}.$

The $\tilde{h}_m$, $2 \le m \le N-2$, form an $(N-3)$-dimensional multiplet of the Möbius algebra, *i.e.* a representation of 'Möbius spin' $\frac{1}{2}N-2$. The equations $\tilde{h}_m(z_1,\ldots,z_n) = \sum_{\substack{U\subset A\\ |U|=m}} k_U^2 z_U = 0$ determine a discrete set of points (up to Möbius invariance).

Amplitudes in terms of Polynomial Constraints

$$\mathcal{A}_N = \oint_{\mathcal{O}} \Psi_N(z,k) \prod_{m=1}^{N-3} \frac{1}{h_m(z,k)} \prod_{2 \le a < b \le N-1} (z_a - z_b) \prod_{a=2}^{N-2} \frac{z_a dz_{a+1}}{(z_a - z_{a+1})^2}.$$

$$h_m = \lim_{z_1 \to \infty} \frac{\widetilde{h}_{m+1}}{z_1} = \frac{1}{m!} \sum_{\substack{a_1, a_2, \ldots, a_m \neq 1, N \\ a_i \text{ uneq.}}} k^2_{1 a_1 \ldots a_m} z_{a_1} z_{a_2} \ldots z_{a_m}, \qquad 1 \le m \le N-3,$$

The $N-3$ polynomial equations $h_m = 0$, of order m,
linear in each z_a individually,
are equivalent to the Scattering Equations $f_a = \sum_b \frac{k_a \cdot k_b}{z_a - z_b} = 0$.

By Bézout's theorem, they determine $(N-3)!$ solutions for the $(N-3)$ ratios $z_2/z_{N-1}, z_3/z_{N-1}, \ldots, z_{N-2}/z_{N-1}$.

$$k_i^2 = 0, \qquad k_{12\ldots a}^2 = (k_1 + k_2 + \ldots k_a)^2 = 2k_1 \cdot k_2 + 2k_1 \cdot k_3 + \ldots,$$

The Scattering Equations:

$$N = 4 \qquad h_1 = k_{12}^2\, z_2 + k_{13}^2\, z_3 = 0,$$

$$N = 5 \qquad \begin{aligned} h_1 &= k_{12}^2\, z_2 + k_{13}^2\, z_3 + k_{14}^2\, z_4 = 0, \\ h_2 &= k_{123}^2\, z_2\, z_3 + k_{124}^2\, z_2\, z_4 + k_{134}^2\, z_3\, z_4 = 0, \end{aligned}$$

$$N = 6 \qquad \begin{aligned} h_1 &= k_{12}^2\, z_2 + k_{13}^2\, z_3 + k_{14}^2\, z_4 + k_{15}^2\, z_5 = 0, \\ h_2 &= k_{123}^2\, z_2\, z_3 + k_{124}^2\, z_2\, z_4 + k_{125}^2\, z_2\, z_5 \\ &\quad + k_{134}^2\, z_3\, z_4 + k_{135}^2\, z_3\, z_5 + k_{145}^2\, z_4\, z_5 = 0, \\ h_3 &= k_{1234}^2\, z_2\, z_3\, z_4 + k_{1235}^2\, z_2\, z_3\, z_5 + k_{1345}^2\, z_3\, z_4\, z_5 = 0. \end{aligned}$$

$$N \qquad h_1, h_2, \ldots, h_{N-3} = 0, \qquad z_2, z_3, \ldots, z_{N-2}, z_{N-1}.$$

Amplitudes as Algebraic Objects attached to a Variety

$$\mathcal{A}_{\mathcal{N}} = \sum_{\text{solutions}} \frac{\Psi_N(z,k)}{J(z,k)} \prod_{2\leq a<b\leq N-1} (z_a - z_b) \prod_{a=2}^{N-2} \frac{z_a dz_a}{(z_a - z_{a+1})^2}$$

$$J(z,k) = \det \left[\frac{\partial h_m}{\partial z_a} \right]_{\substack{1\leq m\leq N-3 \\ 2\leq a\leq N-2}} .$$

The integrals are somewhat symbolic, just sums over the solutions of the Scattering Equations, and hence rational functions of the Mandelstam variables.

For the ring of polynomials in $CP^{N-3}(z_2, \ldots, z_{N-2})$, consider the ideal associated with the $h_1, \ldots, h_{N-3}$ polynomials. The equations $h_m = 0$ define a projective variety, which is a set of $(N-3)!$ points.

The goal is to understand the amplitudes in terms of natural algebraic objects attached to the variety in CP^{N-3} described by the Scattering Equations.

To solve $h_m(z,k) = 0, \quad 1 \leq m \leq N-3$,

we will eliminate z_a, $2 \leq a \leq N-3$,

in terms of $u = z_{N-2}$ and $v = z_{N-1}$, to give

a single variable polynomial equation of order $(N-3)!$ in u/v,
whose roots determine the solutions of the Scattering Equations.

Linear equations determine $z_2, \ldots, z_{N-3}$ from u/v.

Solving the Scattering Equations

$N = 4$

$$h_1 = k_{12}^2 z_2 + k_{13}^2 z_3 = 0, \qquad z_2/z_3 = -k_{13}^2/k_{12}^2 = -k_1 \cdot k_3/k_1 \cdot k_2.$$

$N = 5$

$$\sigma_{ab\ldots} \equiv (k_1 + k_a + k_b + \ldots)^2$$

$$z_2 = x,\ z_3 = u,\ z_4 = v$$

$$h_1 = \sigma_2 x + \sigma_3 u + \sigma_4 v = 0,$$
$$h_2 = \sigma_{23} xu + \sigma_{24} xv + \sigma_{34} uv = 0,$$

eliminating x yields a quadratic equation for u/v.
This can be written as

$$0 = \begin{vmatrix} \sigma_3 u + \sigma_4 v & \sigma_2 \\ \sigma_{34} uv & \sigma_{23} u + \sigma_{24} v \end{vmatrix} = \begin{vmatrix} h_1 & \frac{\partial h_1}{\partial x} \\ h_2 & \frac{\partial h_2}{\partial x} \end{vmatrix} = \Delta_5,$$

which is independent of x.

Another way to establish that Δ_5 is independent of x

Let $\Delta_5 = 0$ be the condition on u, v such that $h_1 = 0$, $h_2 = 0$ have a common solution for some x.

If $\Delta_5 = \begin{vmatrix} h_1 & \frac{\partial h_1}{\partial x} \\ h_2 & \frac{\partial h_2}{\partial x} \end{vmatrix} = 0$ for some $x = x_0$, then there exists a solution ξ such that

$$h_1(x_0 + \xi, u, v) = h_1(x_0, u, v) + \xi \frac{\partial h_1}{\partial x}(x_0, u, v) = 0,$$
$$h_2(x_0 + \xi, u, v) = h_2(x_0, u, v) + \xi \frac{\partial h_2}{\partial x}(x_0, u, v) = 0,$$

since h_m is linear in each of the variables x, u, v separately.

Then Δ_5 is independent of x.

$N = 6$ write $(x, y, u, v) = (z_2, z_3, z_4, z_5)$

$$h_1 = \sigma_2 x + \sigma_3 y + \sigma_4 u + \sigma_5 v = 0,$$
$$h_2 = \sigma_{23} xy + \sigma_{24} xu + \sigma_{34} yu + \sigma_{25} xv + \sigma_{35} yv + \sigma_{45} uv = 0,$$
$$h_3 = \sigma_{234} xyu + \sigma_{235} xyv + \sigma_{245} xuv + \sigma_{345} yuv = 0,$$

eliminating x, y yields a sextic equation for u/v.
This can be written

$$\Delta_6 = \begin{vmatrix} h_1 & h_1^y & h_1^x & h_1^{xy} & 0 & 0 \\ h_2 & h_2^y & h_2^x & h_2^{xy} & 0 & 0 \\ h_3 & h_3^y & h_3^x & h_3^{xy} & 0 & 0 \\ 0 & 0 & h_1 & h_1^y & h_1^x & h_1^{xy} \\ 0 & 0 & h_2 & h_2^y & h_2^x & h_2^{xy} \\ 0 & 0 & h_3 & h_3^y & h_3^x & h_3^{xy} \end{vmatrix} = 0,$$

$$h_m^x = \frac{\partial h_m}{\partial x}, \qquad h_m^{xy} = \frac{\partial^2 h_m}{\partial x \partial y}, \quad \text{etc.} \qquad \text{where } \frac{\partial \Delta_6}{\partial x} = \frac{\partial \Delta_6}{\partial y} = 0$$

Elimination theory developed by Sylvester and Cayley

Supplement $h_1 = h_2 = h_3 = 0$ with $xh_1 = xh_2 = xh_3 = 0$, providing 6 linear relations between $1, x, y, xy, x^2, x^2y$,

$$h_m = a_m + b_m y + c_m x + d_m xy = 0,$$
$$xh_m = a_m x + b_m xy + c_m x^2 + d_m x^2 y = 0,$$

The condition of their consistency is

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 & 0 & 0 \\ a_2 & b_2 & c_2 & d_2 & 0 & 0 \\ a_3 & b_3 & c_3 & c_3 & 0 & 0 \\ 0 & 0 & a_1 & b_1 & c_1 & d_1 \\ 0 & 0 & a_2 & b_2 & c_2 & d_2 \\ 0 & 0 & a_3 & b_3 & c_3 & d_3 \end{vmatrix} = 0,$$

Elimination theory developed by Sylvester and Cayley

Supplement $h_1 = h_2 = h_3 = 0$ with $xh_1 = xh_2 = xh_3 = 0$, providing 6 linear relations between $1, x, y, xy, x^2, x^2y$,

$$h_m = a_m + b_m y + c_m x + d_m xy = 0,$$
$$xh_m = a_m x + b_m xy + c_m x^2 + d_m x^2 y = 0,$$

The condition of their consistency is equal to

$$\Delta_6 = \begin{vmatrix} h_1 & h_1^y & h_1^x & h_1^{xy} & 0 & 0 \\ h_2 & h_2^y & h_2^x & h_2^{xy} & 0 & 0 \\ h_3 & h_3^y & h_3^x & h_3^{xy} & 0 & 0 \\ 0 & 0 & h_1 & h_1^y & h_1^x & h_1^{xy} \\ 0 & 0 & h_2 & h_2^y & h_2^x & h_2^{xy} \\ 0 & 0 & h_3 & h_3^y & h_3^x & h_3^{xy} \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 & d_1 & 0 & 0 \\ a_2 & b_2 & c_2 & d_2 & 0 & 0 \\ a_3 & b_3 & c_3 & c_3 & 0 & 0 \\ 0 & 0 & a_1 & b_1 & c_1 & d_1 \\ 0 & 0 & a_2 & b_2 & c_2 & d_2 \\ 0 & 0 & a_3 & b_3 & c_3 & d_3 \end{vmatrix} = 0,$$

since the left determinant is independent of x, y.

As before, the independence of Δ_6 from x, y can be established by noting that $\Delta_6 = 0$ is also the condition for the existence of ξ, η such that

$$h_m(x+\xi, y+\eta, u, v) = h_m + \xi h_m^x + \eta h_m^y + \eta\xi h_m^{xy} = 0,$$
$$\xi h_m(x+\xi, y+\eta, u, v) = \xi h_m + \xi^2 h_m^x + \xi\eta h_m^y + \xi^2\eta h_m^{xy} = 0.$$

So $\Delta_6 = 0$ provides the condition on u, v for a common solution $h_1 = h_2 = h_3 = 0$ independent of x, y.

$$\begin{pmatrix} h_1 & h_1^y & h_1^x & h_1^{xy} & 0 & 0 \\ h_2 & h_2^y & h_2^x & h_2^{xy} & 0 & 0 \\ h_3 & h_3^y & h_3^x & h_3^{xy} & 0 & 0 \\ 0 & 0 & h_1 & h_1^y & h_1^x & h_1^{xy} \\ 0 & 0 & h_2 & h_2^y & h_2^x & h_2^{xy} \\ 0 & 0 & h_3 & h_3^y & h_3^x & h_3^{xy} \end{pmatrix} \begin{pmatrix} 1 \\ \eta \\ \xi \\ \eta\xi \\ \xi^2 \\ \xi^2\eta \end{pmatrix} = 0.$$

Then compute x, y in terms of u, v from linear relations

Use the null vector to find

$$\begin{pmatrix} h_2 & h_2^y & h_2^{xy} & 0 & 0 \\ h_3 & h_3^y & h_3^{xy} & 0 & 0 \\ 0 & 0 & h_1^y & h_1^x & h_1^{xy} \\ 0 & 0 & h_2^y & h_2^x & h_2^{xy} \\ 0 & 0 & h_3^y & h_3^x & h_3^{xy} \end{pmatrix} \begin{pmatrix} 1 \\ \eta \\ \eta\xi \\ \xi^2 \\ \xi^2\eta \end{pmatrix} = - \begin{pmatrix} h_2^x \\ h_3^x \\ h_1 \\ h_2 \\ h_3 \end{pmatrix} \xi,$$

then

$$\eta = \frac{\xi\eta}{\xi} = - \begin{vmatrix} h_2 & h_2^y & h_2^{xy} & 0 & 0 \\ h_3 & h_3^y & h_3^{xy} & 0 & 0 \\ 0 & 0 & h_1^y & h_1^x & h_1^{xy} \\ 0 & 0 & h_2^y & h_2^x & h_2^{xy} \\ 0 & 0 & h_3^y & h_3^x & h_3^{xy} \end{vmatrix}^{-1} \begin{vmatrix} h_2 & h_2^y & 0 & 0 & 0 \\ h_3 & h_3^y & 0 & 0 & 0 \\ 0 & 0 & h_1 & h_1^x & h_1^{xy} \\ 0 & 0 & h_2 & h_2^x & h_2^{xy} \\ 0 & 0 & h_3 & {h_3}^x & h_3^{xy} \end{vmatrix}.$$

For $\eta = 0$, y satisfies $h_m(x, y, u, v) = 0$ for some x.

So compute y in terms of u, v from the linear relation

$$\eta = \frac{\xi\eta}{\xi} = -\begin{vmatrix} h_1^y & h_1^x & h_1^{xy} \\ h_2^y & h_2^x & h_2^{xy} \\ h_3^y & h_3^x & h_3^{xy} \end{vmatrix}^{-1} \begin{vmatrix} h_1 & h_1^x & h_1^{xy} \\ h_2 & h_2^x & h_2^{xy} \\ h_3 & {h_3}^x & h_3^{xy} \end{vmatrix} = 0.$$

Linear in y, independent of x.

Notation:

$$h \equiv \begin{pmatrix} h_1 \\ h_2 \\ h_3 \end{pmatrix}, \qquad \begin{vmatrix} h & h^x & h^{xy} \end{vmatrix} = 0.$$

N=7

Writing $(z_2, z_3, z_4, z_5, z_6) = (x, y, z, u, v)$ we will eliminate x, y, z to obtain a single variable equation for u/v of order $(N-3)! = 24$, using the 24 equations

$$h_m = yh_m = xh_m = xyh_m = x^2 h_m = x^2 yh_m = 0, \qquad 1 \le m \le 4$$

providing linear relations between the 24 monomials

$$x^p y^q z^r, \qquad 0 \le p \le 3,\ 0 \le q \le 2,\ 0 \le r \le 1,$$

with

$$h_m = a_m + b_m z + c_m y + d_m x + e_m yz + f_m xz + g_m xy + j_m xyz.$$

N=7

Writing $(z_2, z_3, z_4, z_5, z_6) = (x, y, z, u, v)$ we will eliminate x, y, z to obtain a single variable equation for u/v of order $(N-3)! = 24$, using the 24 equations

$$h_m = yh_m = xh_m = xyh_m = x^2 h_m = x^2 yh_m = 0, \qquad 1 \leq m \leq 4$$
$$C_2 = \{1, x, y, xy, x^2, x^2 y\},$$

providing linear relations between the 24 monomials

$$C_3 = \left\{x^p y^q z^r, \qquad 0 \leq p \leq 3,\, 0 \leq q \leq 2,\, 0 \leq r \leq 1\right\},$$

with

$$h_m = a_m + b_m z + c_m y + d_m x + e_m yz + f_m xz + g_m xy + j_m xyz,$$
$$B_3 = x^m y^n z^p, \qquad m, n, p = 0, 1.$$

$\Delta_7 = |M_7| = 0 =$

$$\begin{vmatrix}
h & h^z & h^y & h^{yz} & 0 & 0 & h^x & h^{xz} & h^{xy} & h^{xyz} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & h & h^z & h^y & h^{yz} & 0 & 0 & h^x & h^{xz} & h^{xy} & h^{xyz} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & h & h^z & h^y & h^{yz} & 0 & 0 & h^x & h^{xz} & h^{xy} & h^{xyz} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & h & h^z & h^y & h^{yz} & 0 & 0 & h^x & h^{xz} & h^{xy} & h^{xyz} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & h & h^z & h^y & h^{yz} & 0 & 0 & h^x & h^{xz} & h^{xy} & h^{xyz} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & h & h^z & h^y & h^{yz} & 0 & 0 & h^x & h^{xz} & h^{xy} & h^{xyz}
\end{vmatrix}$$

where $h_m^{xyz} = \partial_x \partial_y \partial_z h_m, \quad h = \begin{pmatrix} h_1 \\ h_2 \\ h_3 \\ h_4 \end{pmatrix}.$

Δ_7 vanishes for $h_m = 0$, and is independent of x, y, z.

The rows of M_7 are labeled by $\alpha \in m, C_2$, the columns by $\beta \in C_3$.
The non-zero entries are $M_{m\alpha,\beta} = h_m^\gamma$ if $\beta = \alpha\gamma, \quad \gamma \in B_3$.
$\deg M_{m\alpha,\beta} = m + \deg \alpha + \deg \beta$,
$\deg \Delta_7 = \sum_{m,C_2}(m + \deg \alpha) + \sum_{C_4} \deg \beta = 24.$

The coefficient of v^{24} in Δ_7 is
$h_1^6 (h_2^z)^2 (h_2^y)^2 (h_2^x)^2 (h_3^{yz})^2 (h_3^{zx})^2 (h_3^{xy})^2 (h_4^{xyz})^6 = \sigma_6^6 \sigma_{26}^2 \sigma_{36}^2 \sigma_{46}^2 \sigma_{236}^2 \sigma_{246}^2 \sigma_{346}^2 \sigma_{2346}^2.$

General N

$(N-3)!$ relations $h_m C_{N-5} = 0$ (labeling rows m,α)
between the $(N-3)!$ variables C_{N-4} (labeling columns β).

$$C_M = \{ \prod_{a=1}^{M} x_a^{m_a} : \ 0 \le m_a \le M - a + 1, \ 1 \le a \le M \}$$

$$B_M = \{ \prod_{a=1}^{M} x_a^{m_a} : \ 0 \le m_a \le 1, \ 1 \le a \le M \}$$

$$\begin{aligned} M_{m\alpha,\beta} &= h_m^{\gamma} \qquad \text{if } \beta = \alpha\gamma, \quad \gamma \in B_{N-4}, \\ &= 0 \qquad \text{if } \beta \notin \alpha B_{N-4}. \end{aligned}$$

$$\Delta_N = \det M = 0.$$

$$\deg M_{m\alpha,\beta} = m + \deg \alpha + \deg \beta,$$

$$\deg \Delta_N = \sum_{m=1}^{N-3} \sum_{\alpha \in C_{N-5}} (m + \deg \alpha) - \sum_{\beta \in C_{N-4}} \deg \beta = (N-3)!$$

Since no element of M is more than linear in v, the term $v^{(N-3)!}$ in $\det M$ must come from the product of linear factors $u^0 v^1$.

The element $M_{m\alpha,\beta}$ is of degree one when $m - \deg\gamma = 1$.

The coefficient of $v^{(N-3)!}$ contains

$$\prod_{\gamma \in B_{N-4}} \left[h_m^\gamma\right]^{n_\gamma},$$

where $m = \deg\gamma + 1, \qquad n_\gamma = (N - 4 - \deg\gamma)!(\deg\gamma)!$

Summary

The scattering equations can be reformulated as polynomial equations that are linear in the variables z_a separately. Using Möbius invariance, the polynomials are reduced to $(N-3)$ equations in $(N-3)$ variables.

Facilitated by this linearity, elimination theory is used to construct a polynomial equation of degree $(N-3)!$ for the single variable z_{N-2}/z_{N-1}, determining the (N-3)! solutions expected from Bézout's theorem. Linear relations relate the remaining variables to the single variable.

For the $(N-3)$ equations $h_m(z_2, \ldots, z_{N-2}; k_1, \ldots k_{N-1}) = 0$, the $(N-3)!$ solutions $z_a(\mathbf{k})$ in CP^{N-3} define a set of points forming the variety of the ideal $< h_1, h_2, \ldots h_{N-3} >$. The goal is to understand the N-point scattering amplitudes, which appear as rational functions of the kinematic invariants, as natural algebraic objects attached to this variety.