Adaptive Unitarity and Magnus Exponential for Scattering Amplitudes

MHV @ 30, FermiLab
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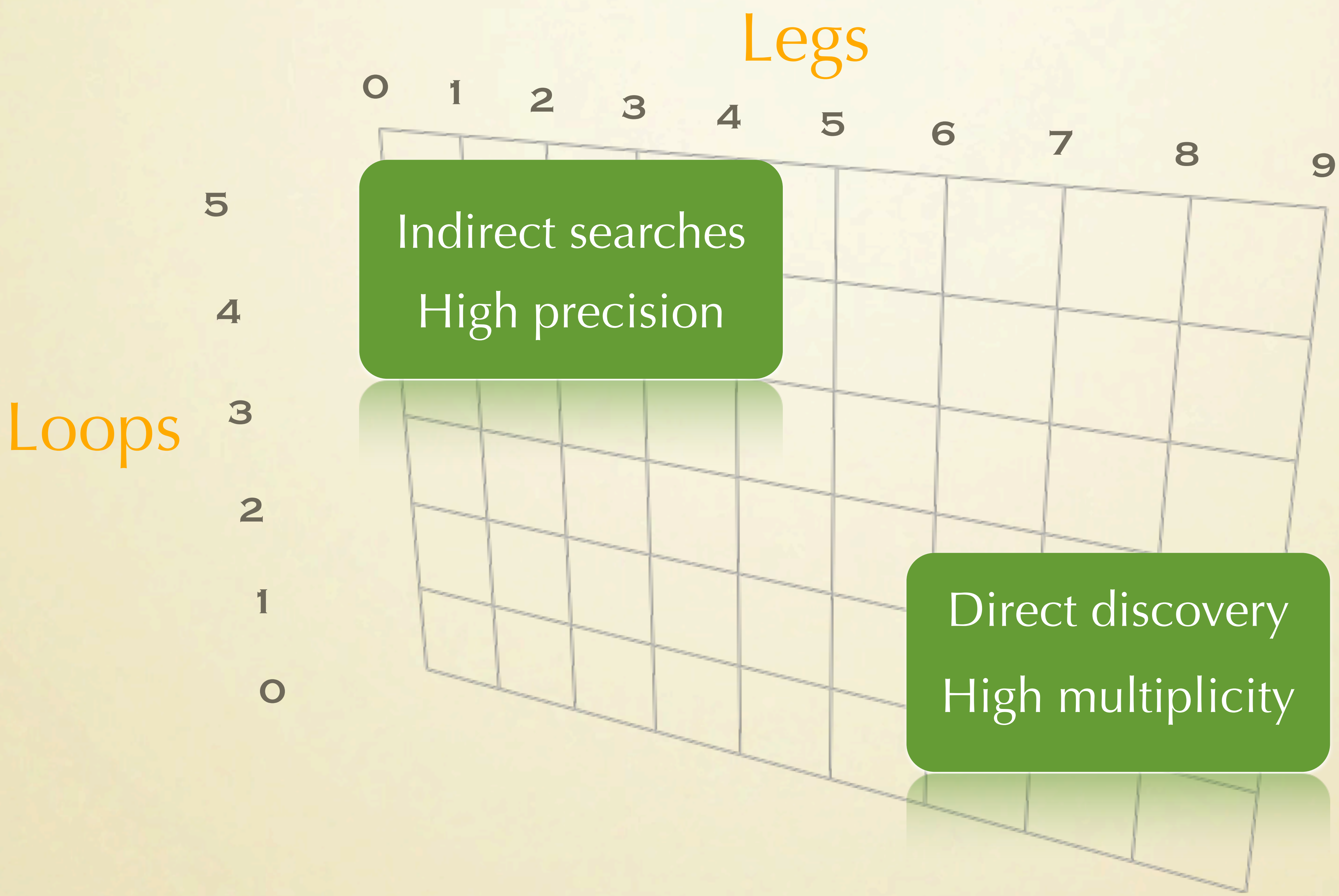
Motivation

- Amplitudes & Phenomenology
  - masses do matter
  - non-planar diagrams may contribute
  - integrals diverge
  - from the beauty of simple formulas (in special kinematics)
    to the beauty of the structures (in arbitrary kinematics)

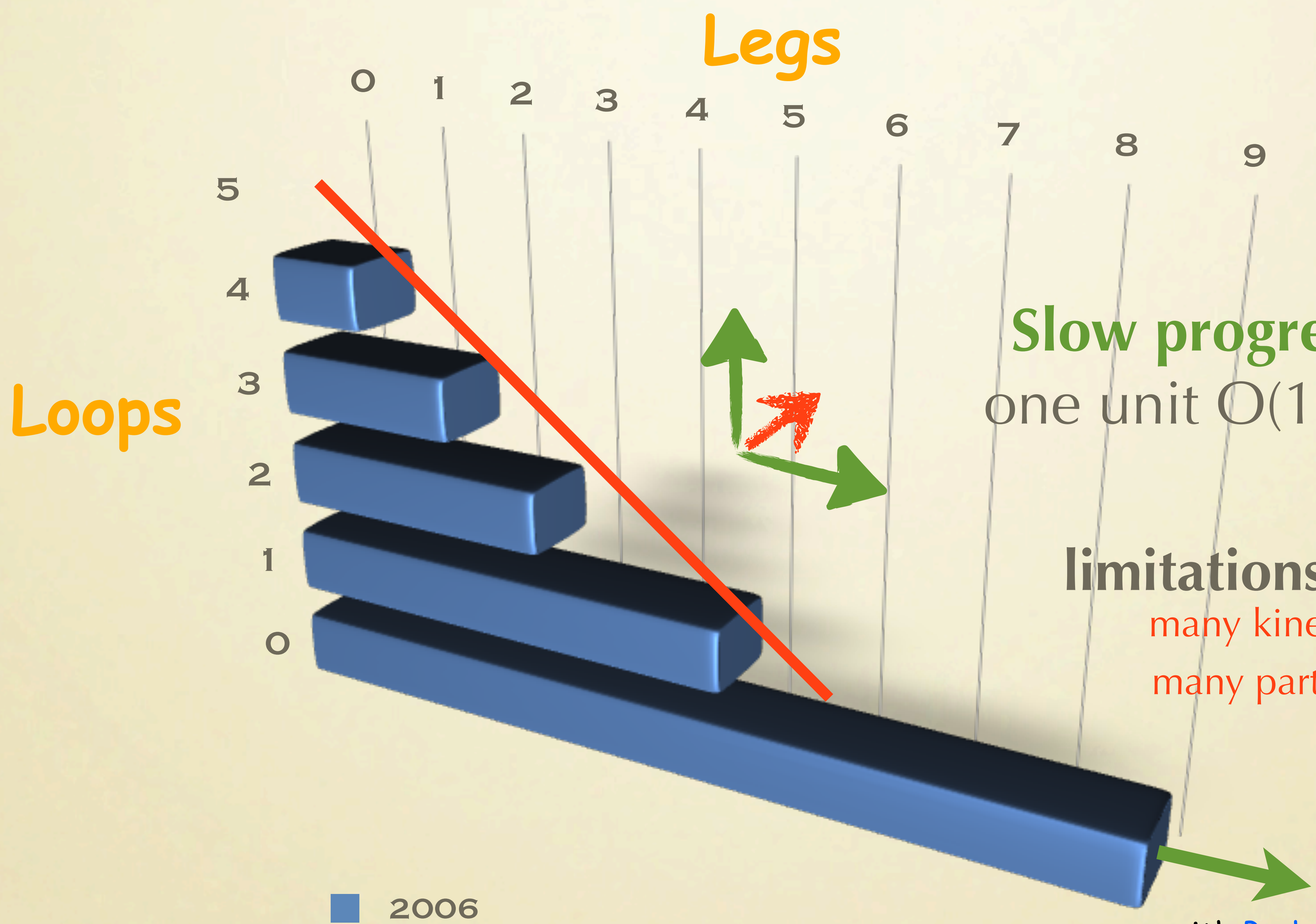
Path

- Multiloop Integrand Decomposition: exploiting dimensional regularisation
- Magnus Series for Master Integrals

High Energy Physics Goals: Loops vs Legs



Complexity: Loops vs Legs

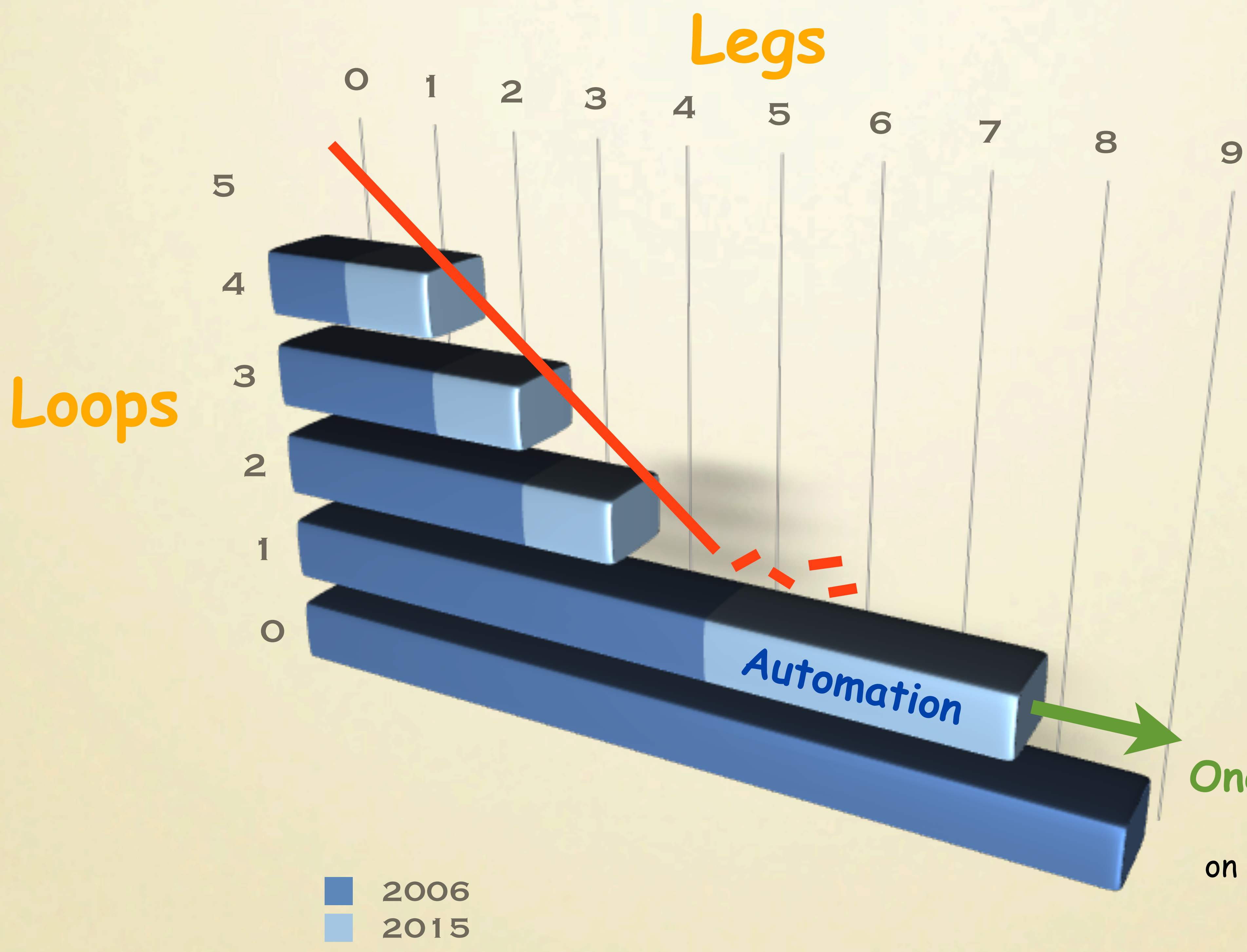


Slow progress:
one unit O(10ys)

limitations:
many kinematic **invariants**
many particle **masses**

with Parke and Taylor
in good company up to ∞

Complexity: Loops vs Legs



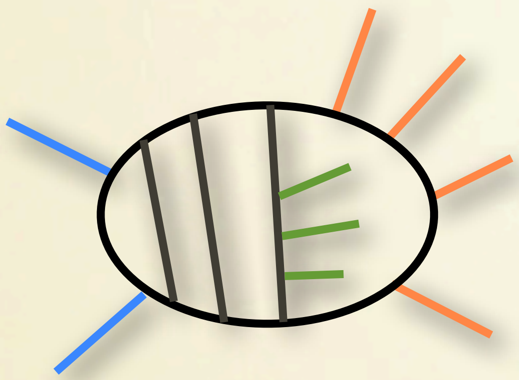
One-Loop Revolution

Dramatic impact
on Collider Phenomenology

>> Kunszt, Kosower

Why is it all that difficult?

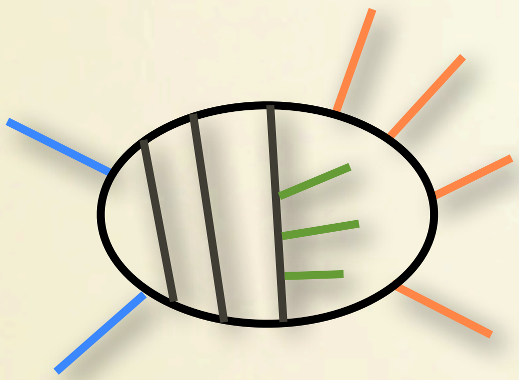
Feynman Diagrams ~ The realm of **Integral Calculus**



$$\sim \int dx \int dy \int dz \ \ldots \ f(x, y, z, \ldots)$$

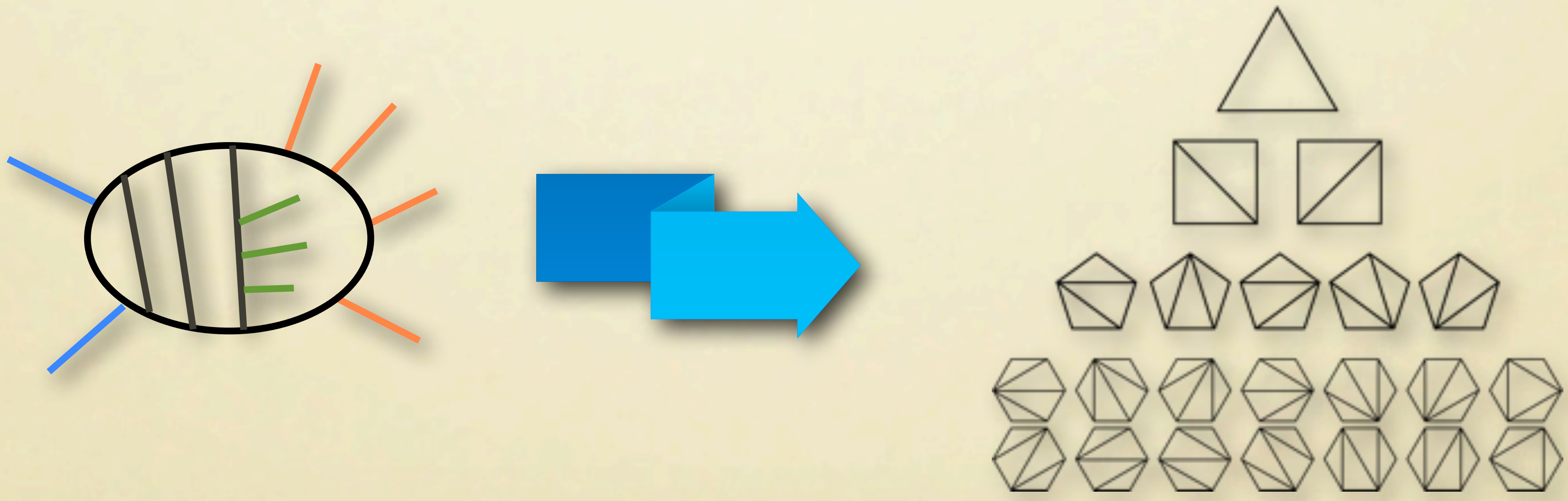
Why is it all that difficult?

Feynman Diagrams ~ The realm of **Integral Calculus**



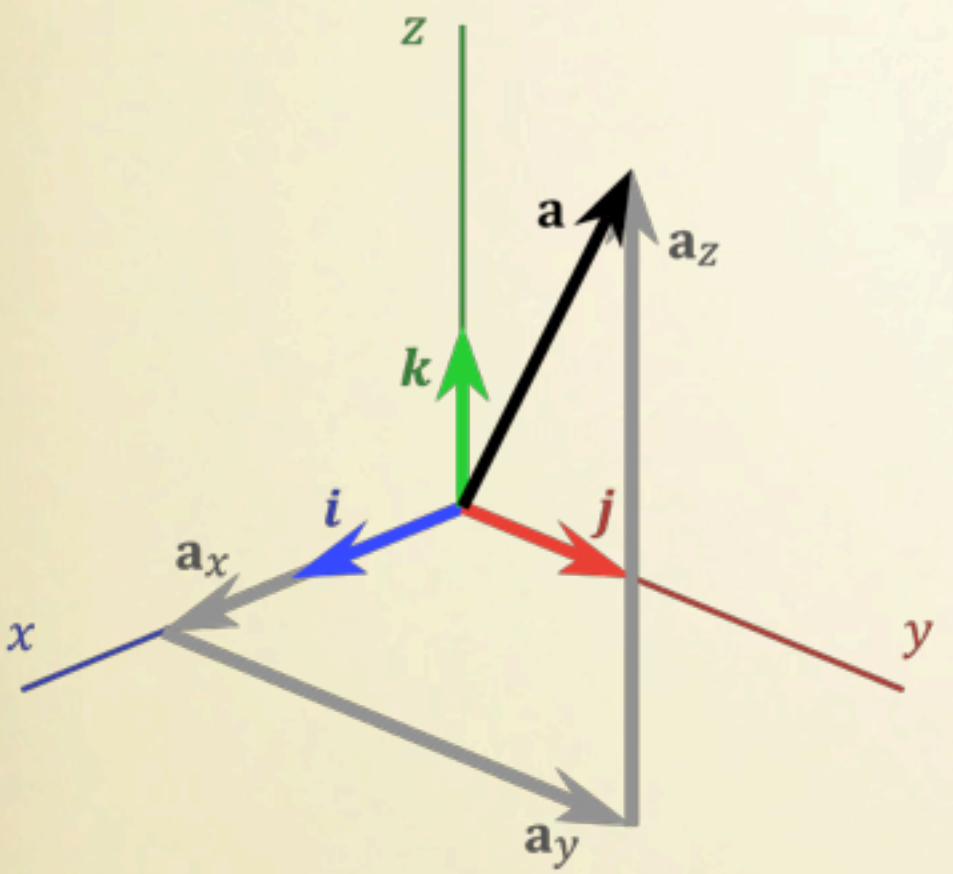
$$\sim \int dx \int dy \int dz \; \ldots \; f(x, y, z, \ldots)$$

Turning **Integral Calculus** into an **Algebraic Problem**



Amplitudes Decomposition:

the algebraic way



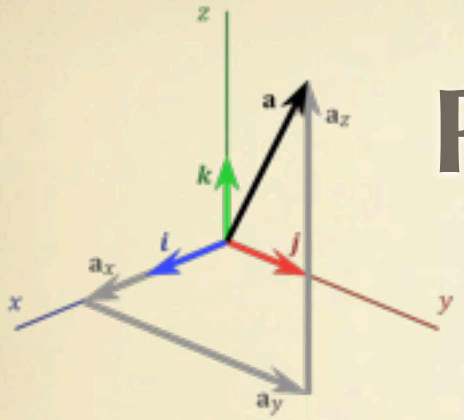
$$\mathbf{a} = a_x\,\mathbf{i} + a_y\,\mathbf{j} + a_z\,\mathbf{k}$$

- Basis: {i j k}
- Scalar product/Projection: to extract the components

$$a_x = \mathbf{a}.\mathbf{i}$$

$$a_y = \mathbf{a}.\mathbf{j}$$

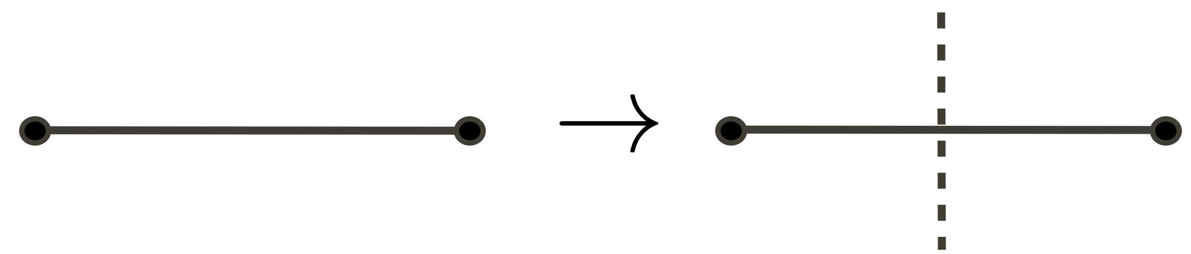
$$a_z = \mathbf{a}.\mathbf{k}$$



Projections :: On-Shell Cut-Conditions

vanishing denominators

$$\frac{1}{p^2 - m^2 - i0} \to \delta(p^2 - m^2)$$



Completeness Relations: cutting "1"

- the richness of factorization

$$i\,(\text{-}i) = 1$$

$$\sum_n |\,\psi_{\mathrm{n}}\,\rangle\,\langle\,\psi_{\mathrm{n}}| \;=\; \mathbb{1}$$

$$(p^2 - m^2) = (\not{p} - m)(\not{p} + m)$$

$$\varepsilon^{\mu\nu} = \varepsilon^{\mu}\varepsilon^{\nu}$$

Completeness Relations: cutting "1"

- the richness of factorization

SuperGravity @ 40
MHV @ 30
TASI lectures @ 20

Integrand-Reduction @10

unitarity at integrand level

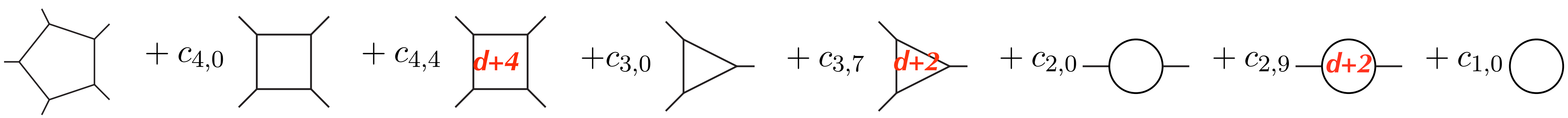
Ossola Papadopoulos Pittau (2006)
Ellis Giele Kunszt Melnikov (2007)
Ossola & ***P.M.*** (2011)
Badger, Frellesvig, Zhang (2011)
Zhang (2012)
Mirabella, Ossola, Peraro, & ***P.M.*** (2012)

One-Loop Integrand Decomposition

$$\mathcal{A}_n^{\text{one-loop}} = \int d^{-2\epsilon}\mu \int d^4 q \; A_n(q,\mu^2) \,, \qquad A_n(q,\mu^2) \equiv \frac{\mathcal{N}_n(q,\mu^2)}{\bar{D}_0 \bar{D}_1 \cdots \bar{D}_{n-1}} \qquad \bar{D}_i = (\bar{q}+p_i)^2 - m_i^2 = (q+p_i)^2 - m_i^2 - \mu^2$$

We use a bar to denote objects living in $d = 4 - 2\epsilon$ dimensions $\qquad \bar{\not{q}} = \not{q} + \not{\mu} \,, \quad$ with $\quad \bar{q}^2 = q^2 - \mu^2 \,.$

$\mathcal{A}_n^{\text{one-loop}} = c_{5,0}$ [pentagon] $+ c_{4,0}$ [box] $+ c_{4,4}$ [box, $d+4$] $+ c_{3,0}$ [triangle] $+ c_{3,7}$ [triangle, $d+2$] $+ c_{2,0}$ [bubble] $+ c_{2,9}$ [bubble, $d+2$] $+ c_{1,0}$ [tadpole]



☑ @ the integrand-level

Ossola, Papadopoulos, Pittau

$$A_n(q,\mu^2) \neq \frac{c_{5,0}}{\bar{D}_0\bar{D}_1\bar{D}_2\bar{D}_3\bar{D}_4} + \frac{c_{4,0} + c_{4,4}\mu^4}{\bar{D}_0\bar{D}_1\bar{D}_2\bar{D}_3} + \frac{c_{3,0} + c_{3,7}\mu^2}{\bar{D}_0\bar{D}_1\bar{D}_2} + \frac{c_{2,0} + c_{2,9}\mu^2}{\bar{D}_0\bar{D}_1} + \frac{c_{1,0}}{\bar{D}_0}$$

$$= \frac{c_{5,0} + f_{01234}(q,\mu^2)}{\bar{D}_0\bar{D}_1\bar{D}_2\bar{D}_3\bar{D}_4} + \frac{c_{4,0} + c_{4,4}\mu^4 + f_{0123}(q,\mu^2)}{\bar{D}_0\bar{D}_1\bar{D}_2\bar{D}_3} + \frac{c_{3,0} + c_{3,7}\mu^2 + f_{012}(q,\mu^2)}{\bar{D}_0\bar{D}_1\bar{D}_2} + \frac{c_{2,0} + c_{2,9}\mu^2 + f_{01}(q,\mu^2)}{\bar{D}_0\bar{D}_1} + \frac{c_{1,0} + f_0(q,\mu^2)}{\bar{D}_0}$$

☑ f's are "spurious" ==> integrate to 0 !!!

Improved Integrand Red′n

- **Integrand Reduction**

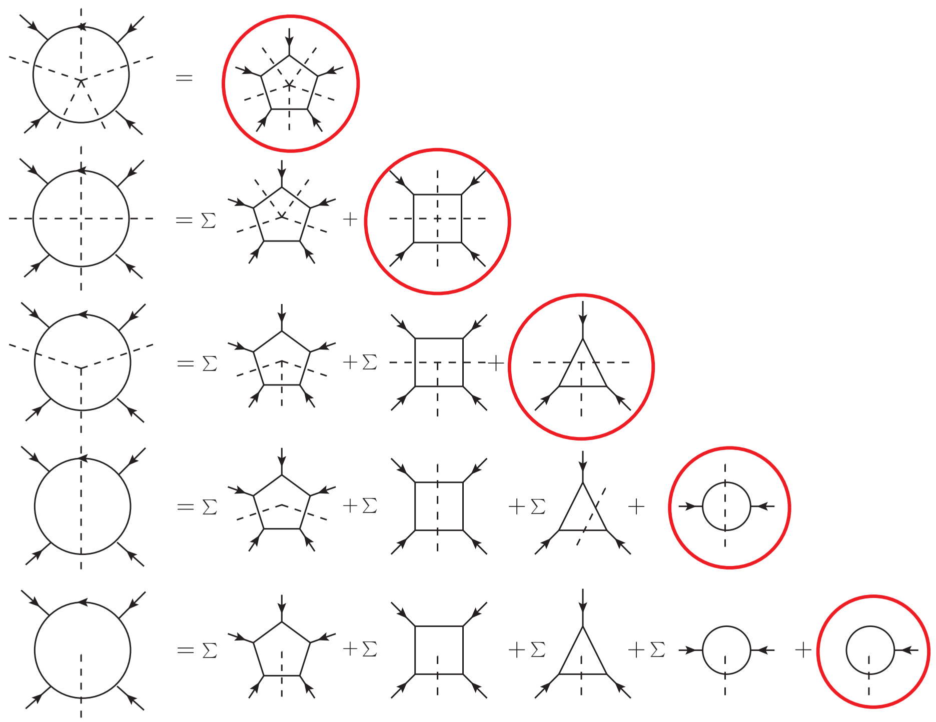
universal

$$\Delta_{i_1\ldots i_m}(q,\mu^2) = \mathrm{Res}_{i_1\ldots i_m}\left\{\frac{\mathcal{N}(q,\mu^2)}{\bar{D}_{i_1}\bar{D}_{i_2}\ldots\bar{D}_{i_n}} - \sum_{k=(m+1)}^{5}\sum_{i_1<i_2<\ldots<i_k}\frac{\Delta_{i_1 i_2\ldots i_k}(q,\mu^2)}{\bar{D}_{i_1}\bar{D}_{i_2}\ldots\bar{D}_{i_k}}\right\}$$

polynomial — non-polynomial — non-polynomial

a + b x + c x^2 + ...

Ossola Papadopoulos Pittau



integrand subtraction required!

Improved Integrand Red'n

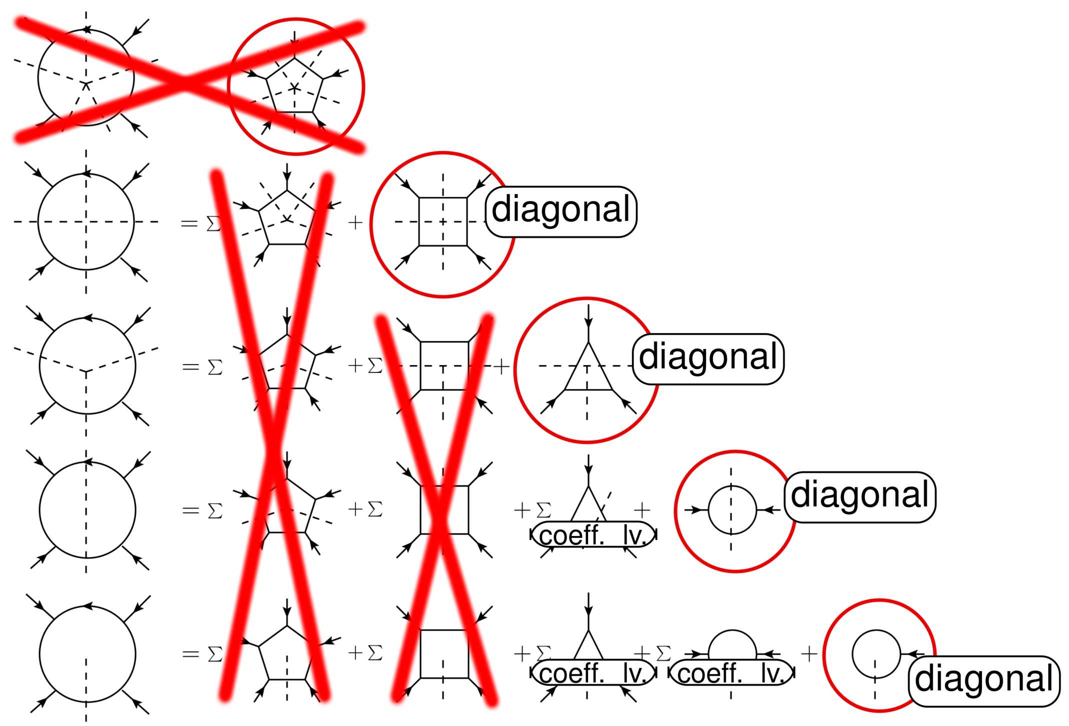
- **Integrand Reduction** with **Laurent series expansion** Forde; Kilgore; Badger;

$$\Delta_{i_1 \ldots i_m}(q,\mu^2) = \mathrm{Res}_{i_1 \ldots i_m} \left\{ \frac{\mathcal{N}(q,\mu^2)}{\bar{D}_{i_1} \bar{D}_{i_2} \ldots \bar{D}_{i_n}} - \sum_{k=(m+1)}^{5} \sum_{i_1 < i_2 < \ldots < i_k} \frac{\Delta_{i_1 i_2 \ldots i_k}(q,\mu^2)}{\bar{D}_{i_1} \bar{D}_{i_2} \ldots \bar{D}_{i_k}} \right\}$$

polynomial: a + b x + c x^2 + ...

∞ polynomial: a′+ b′ x + c′ x^2 + ...

universal ∞ polynomial: a″ + b″ x + c″ x^2 + ...



- **coefficients of MI's :: a = a′+ a″**
- **Laurent series** implemented via ***univariate*** **Polynomial Division**

Mirabella Peraro & ***P.M.*** (2012)

2.2.2 Quintuple cut

The residue of the quintuple-cut, $\bar{D}_i = \ldots = \bar{D}_m = 0$, defined as,

$$\Delta_{ijk\ell m}(\bar{q}) = \text{Res}_{ijk\ell m}\left\{\frac{N(\bar{q})}{\bar{D}_0\cdots\bar{D}_{n-1}}\right\} = c_{5,0}^{(ijk\ell m)}\,\mu^2 \,.$$

2.2.3 Quadruple cut

The residue of the quadruple-cut, $\bar{D}_i = \ldots = \bar{D}_\ell = 0$, defined as,

$$\Delta_{ijk\ell}(\bar{q}) = \text{Res}_{ijk\ell}\left\{\frac{N(\bar{q})}{\bar{D}_0\cdots\bar{D}_{n-1}} - \sum_{i<<m}^{n-1}\frac{\Delta_{ijk\ell m}(\bar{q})}{\bar{D}_i\bar{D}_j\bar{D}_k\bar{D}_\ell\bar{D}_m}\right\} = c_{4,0}^{(ijk\ell)}+c_{4,2}^{(ijk\ell)}\mu^2+c_{4,4}^{(ijk\ell)}\mu^4 -\left(c_{4,1}^{(ijk\ell)}+c_{4,3}^{(ijk\ell)}\,\mu^2\right)\Big[(K_3\cdot e_4)x_4-(K_3\cdot e_3)x_3\Big](e_1\cdot e_2)\,,$$

2.2.4 Triple cut

The residue of the triple-cut, $\bar{D}_i = \bar{D}_j = \bar{D}_k = 0$, defined as,

$$\Delta_{ijk}(\bar{q}) = \text{Res}_{ijk}\left\{\frac{N(\bar{q})}{\bar{D}_0\cdots\bar{D}_{n-1}} - \sum^{n-1}\frac{\Delta_{ijk\ell m}(\bar{q})}{\bar{D}_i\bar{D}_j\bar{D}_k\bar{D}_\ell\bar{D}_m} - \sum^{n-1}\frac{\Delta_{ijk\ell}(\bar{q})}{\bar{D}_i\bar{D}_j\bar{D}_k\bar{D}_\ell}\right\}$$

SAMURAI Ossola Reiter Tramontano ***P.M.*** (2010)

Scattering **AM**plitudes from **U**nitarity-based **R**eduction **A**lgorithm at the **I**ntegrand-level

$$= c_{2,0}^{(ij)} + c_{2,9}^{(ij)}\mu^2 + \left(c_{2,1}^{(ij)}x_1 - c_{2,3}^{(ij)}x_4 - c_{2,5}^{(ij)}x_3\right)(e_1\cdot e_2) + \left(c_{2,2}^{(ij)}x_1^2 + c_{2,4}^{(ij)}x_4^2 + c_{2,6}^{(ij)}x_3^2 - c_{2,7}^{(ij)}x_1x_4 - c_{2,8}^{(ij)}x_1x_3\right)(e_1\cdot e_2)^2 \,.$$

2.2.6 Single cut

The residue of the single-cut, $\bar{D}_i = 0$, defined as,

$$\Delta_i(\bar{q}) = \text{Res}_i\left\{\frac{N(\bar{q})}{\bar{D}_0\cdots\bar{D}_{n-1}} - \sum_{i<<m}^{n-1}\frac{\Delta_{ijk\ell m}(\bar{q})}{\bar{D}_i\bar{D}_j\bar{D}_k\bar{D}_\ell\bar{D}_m} - \sum_{i<<\ell}^{n-1}\frac{\Delta_{ijk\ell}(\bar{q})}{\bar{D}_i\bar{D}_j\bar{D}_k\bar{D}_\ell} - \sum_{i<<k}^{n-1}\frac{\Delta_{ijk}(\bar{q})}{\bar{D}_i\bar{D}_j\bar{D}_k} - \sum_{i<j}^{n-1}\frac{\Delta_{ij}(\bar{q})}{\bar{D}_i\bar{D}_j}\right\}$$

$$= c_{1,0}^{(i)} + \left(c_{1,1}^{(i)}x_2 + c_{1,2}^{(i)}x_1 - c_{1,3}^{(i)}x_4 - c_{1,4}^{(i)}x_3\right)(e_1\cdot e_2) \,.$$

2.2.2 Quintuple cut

The residue of the quintuple-cut, $\bar{D}_i = \ldots = \bar{D}_m = 0$, defined as,

$$\Delta_{ijk\ell m}(\bar{q}) = \text{Res}_{ijk\ell m}\left\{\frac{N(\bar{q})}{\bar{D}_0 \cdots \bar{D}_{n-1}}\right\} = c_{5,0}^{(ijk\ell m)}\ \mu^2 \,.$$

2.2.3 Quadruple cut

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$$\Delta_{ijk\ell}(\bar{q}) = \text{Res}_{ijk\ell}\left\{\frac{N(\bar{q})}{\bar{D}_0 \cdots \bar{D}_{n-1}} - \sum_{i<<m}^{n-1}\frac{\Delta_{ijk\ell m}(\bar{q})}{\bar{D}_i\bar{D}_j\bar{D}_k\bar{D}_\ell\bar{D}_m}\right\} = c_{4,0}^{(ijk\ell)} + c_{4,2}^{(ijk\ell)}\mu^2 + c_{4,4}^{(ijk\ell)}\mu^4 - \left(c_{4,1}^{(ijk\ell)} + c_{4,3}^{(ijk\ell)}\ \mu^2\right)\Big[(K_3\cdot e_4)x_4 - (K_3\cdot e_3)x_3\Big](e_1\cdot e_2)\,,$$

2.2.4 Triple cut

The residue of the triple-cut, $\bar{D}_i = \bar{D}_j = \bar{D}_k = 0$, defined as,

$$\Delta_{ijk}(\bar{q}) = \text{Res}_{ijk}\left\{\frac{N(\bar{q})}{\bar{D}_0 \cdots \bar{D}_{n-1}} - \sum^{n-1}\frac{\Delta_{ijk\ell m}(\bar{q})}{\bar{D}_i\bar{D}_j\bar{D}_k\bar{D}_\ell\bar{D}_m} - \sum^{n-1}\frac{\Delta_{ijk\ell}(\bar{q})}{\bar{D}_i\bar{D}_j\bar{D}_k\bar{D}_\ell}\right\}$$

SAMURAI Ossola Reiter Tramontano ***P.M.*** (2010)

NINJA Mirabella Peraro ***P.M.*** (2013) C++ Peraro (2014)

Integrand decomposition via Laurent expansion

$$= c_{2,0}^{(ij)} + c_{2,9}^{(ij)}\mu^2 + \left(c_{2,1}^{(ij)}x_1 - c_{2,3}^{(ij)}x_4 - c_{2,5}^{(ij)}x_3\right)(e_1\cdot e_2) + \left(c_{2,2}^{(ij)}x_1^2 + c_{2,4}^{(ij)}x_4^2 + c_{2,6}^{(ij)}x_3^2 - c_{2,7}^{(ij)}x_1x_4 - c_{2,8}^{(ij)}x_1x_3\right)(e_1\cdot e_2)^2\,.$$

2.2.6 Single cut

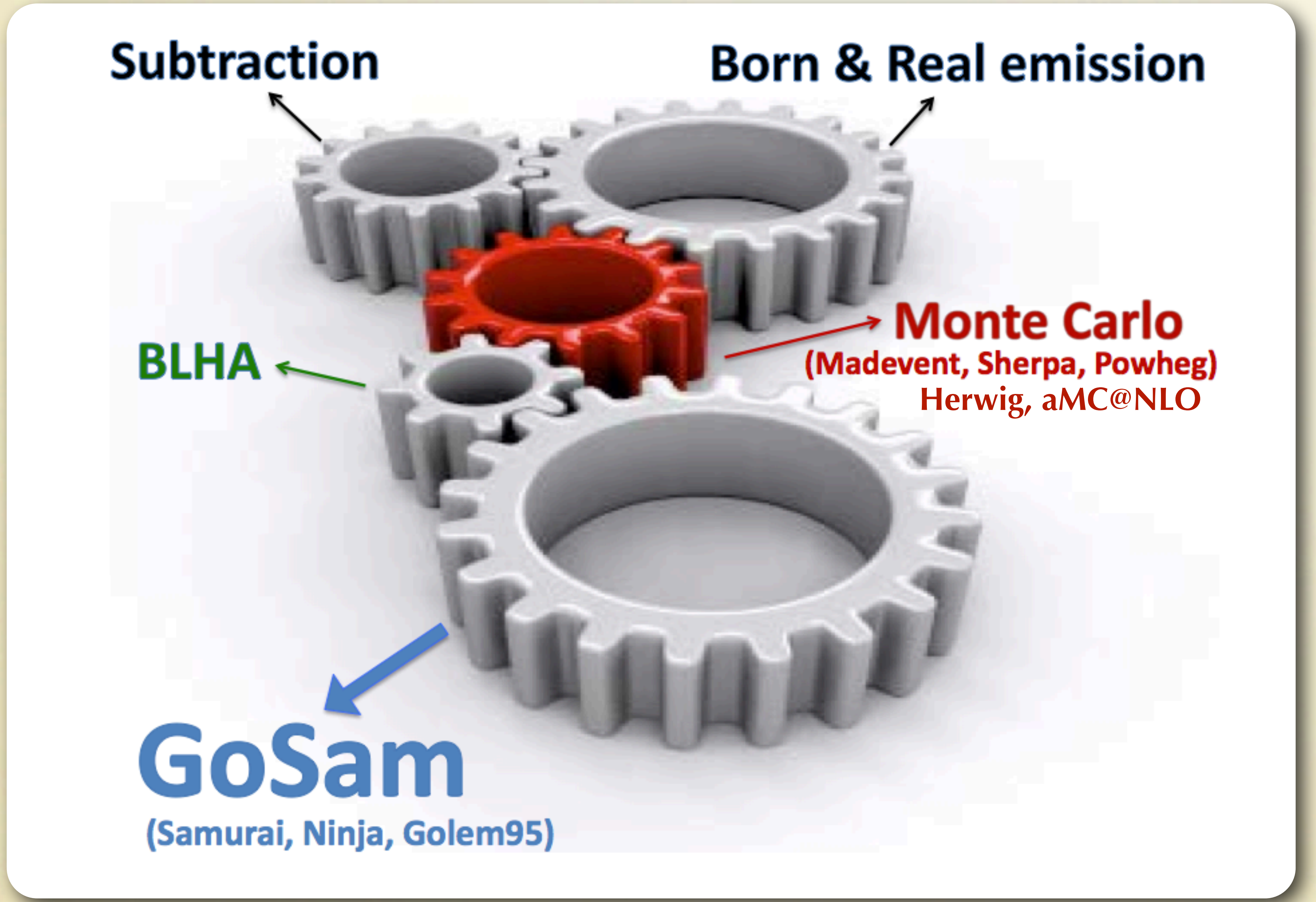
The residue of the single-cut, $\bar{D}_i = 0$, defined as,

$$\Delta_i(\bar{q}) = \text{Res}_i\left\{\frac{N(\bar{q})}{\bar{D}_0 \cdots \bar{D}_{n-1}} - \sum_{i<<m}^{n-1}\frac{\Delta_{ijk\ell m}(\bar{q})}{\bar{D}_i\bar{D}_j\bar{D}_k\bar{D}_\ell\bar{D}_m} - \sum_{i<<\ell}^{n-1}\frac{\Delta_{ijk\ell}(\bar{q})}{\bar{D}_i\bar{D}_j\bar{D}_k\bar{D}_\ell} - \sum_{i<<k}^{n-1}\frac{\Delta_{ijk}(\bar{q})}{\bar{D}_i\bar{D}_j\bar{D}_k} - \sum_{i<j}^{n-1}\frac{\Delta_{ij}(\bar{q})}{\bar{D}_i\bar{D}_j}\right\}$$

$$= c_{1,0}^{(i)} + \left(c_{1,1}^{(i)}x_2 + c_{1,2}^{(i)}x_1 - c_{1,3}^{(i)}x_4 - c_{1,4}^{(i)}x_3\right)(e_1\cdot e_2)\,.$$

The GoSam Project

Cullen van Deurzen Greiner Heinrich Luisoni
Mirabella Ossola Peraro Reichel Schlenk
von Soden-Fraunhofen Tramontano ***P.M.***



MC Interfaces

Beyond SM

EW Physics

Top Physics

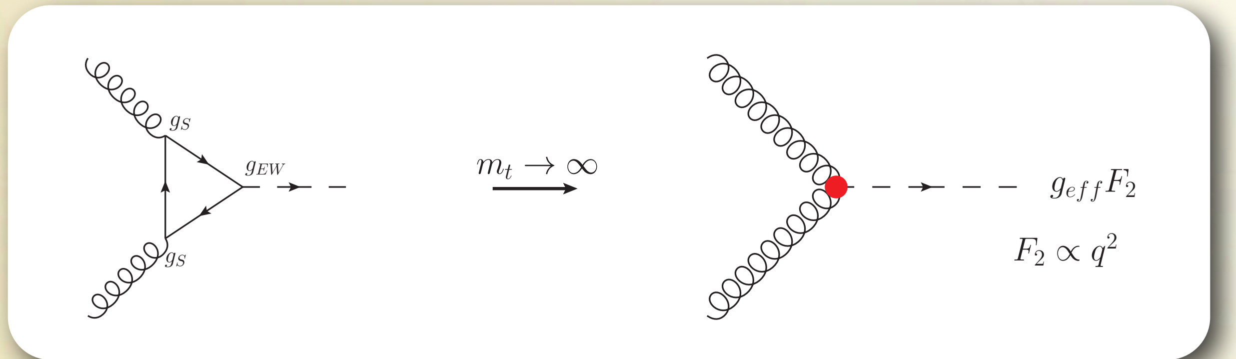
Diphoton and jets

Higgs & Jets

The path to Hjjj @ NLO

Challenges

- *effective Hgg-coupling*:

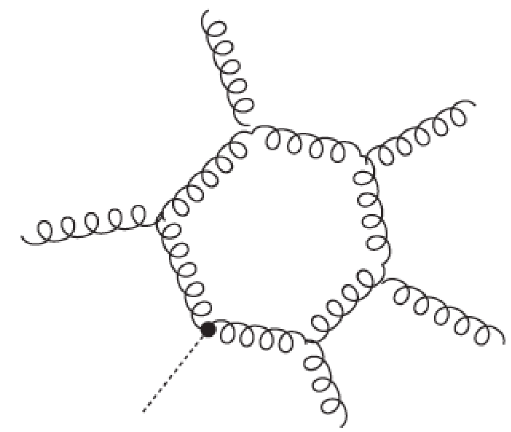


higher rank :: r < n+2

the rank *r* of the numerator can be larger than
the number *n* of denominators

☑ Extending the Polynomial Residues

Mirabella Peraro ***P.M.***



H+0j	**1 NLO**
$gg \to H$	1 NLO
H+1j	**62 NLO**
$qq \to Hqq$	14 NLO
$qg \to Hqg$	48 NLO
H+2j	**926 NLO**
$qq' \to Hqq'$	32 NLO
$qq \to Hqq$	64 NLO
$qg \to Hqg$	179 NLO
$gg \to Hgg$	651 NLO
H+3j	**13179 NLO**
$qq' \to Hqq'g$	467 NLO
$qq \to Hqqg$	868 NLO
$qg \to Hqgg$	2519 NLO
$gg \to Hggg$	9325 NLO

- Over 10,000 diagrams
- Higher-Rank terms
- 60 Rank-7 hexagons

pp --> Hjjj with GoSam

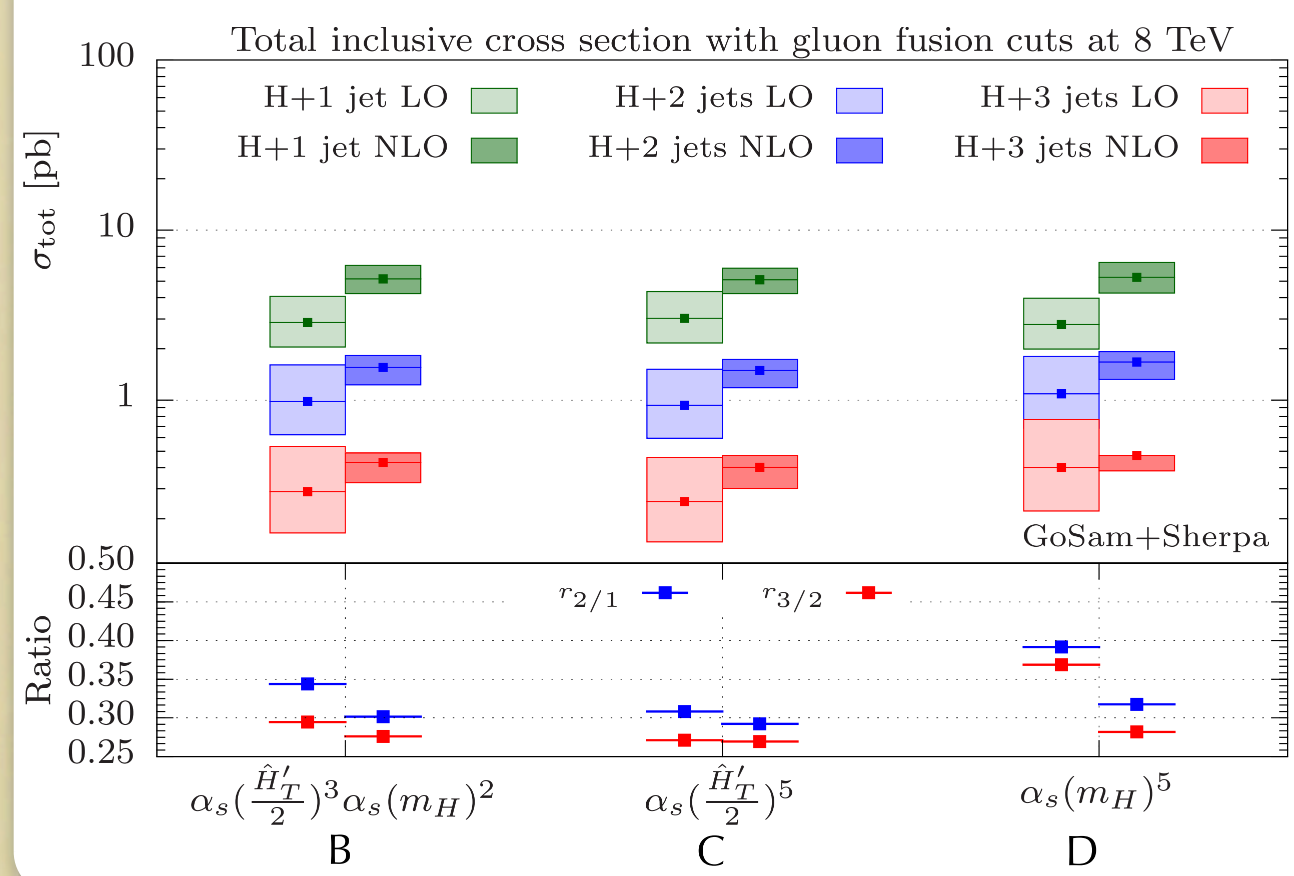
- **Hjj with GoSam + Sherpa (Amegic)** vanDeurzen Greiner Luisoni Mirabella Ossola Peraro vonSodenFraunhofen Tramontano & **P.M.**
- **Hjjj with GoSam + Sherpa + MadGraph4** Cullen VanDeurzen Greiner Luisoni Mirabella Ossola Peraro Tramontano & **P.M.**
- **Hjjj (virtual) with GoSam2.0:** improved reduction **(Ninja)** vanDeurzen Luisoni Mirabella Ossola Peraro & **P.M.**

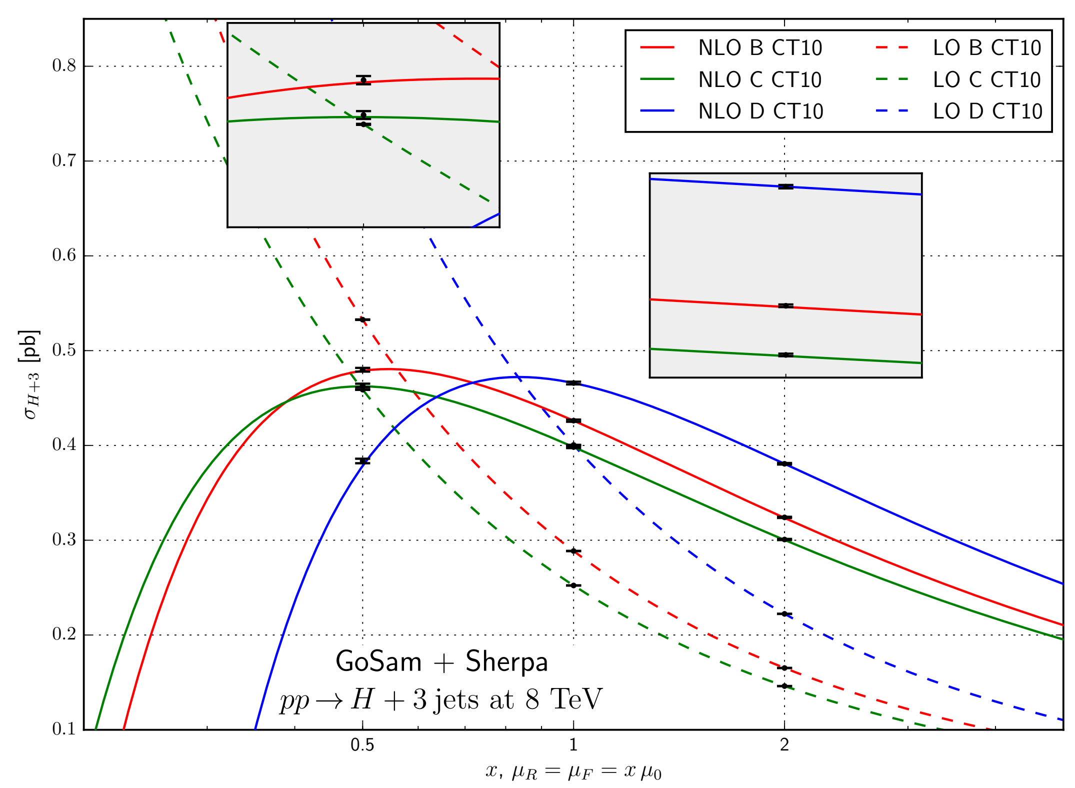
Hj, Hjj, Hjjj with GoSam2.0 + Sherpa (Comix): *a new analysis*

Greiner Hoecke Luisoni Schoenherr Winter Yundin

- Cuts: 8 TeV, anti-kt $R = 0.4$ jets with $p_T > 30$ GeV, $|\eta| < 4.4$
- PDF: CT10nlo for LO, CT10nlo for NLO

$$\hat{H}_T = \sqrt{m_H^2 + p_{T,H}^2} + \sum_i^{partons} p_{T,i}$$





GoSam + Ninja: more app's

Mirabella Peraro ***P.M.*** (2012)
van Deurzen Luisoni Mirabella Ossola Peraro ***P.M.*** (2013)
Peraro (2014)

Benchmarks: GoSam + Ninja			
Process		# NLO diagrams	ms/event
$W+3j$	$d\bar{u} \to \bar{\nu}_e e^- ggg$	1 411	226
$Z+3j$	$d\bar{d} \to e^+ e^- ggg$	2 928	1 911
$ZZZ+1j$	$u\bar{u} \to ZZZg$	915	*12 000
$WWZ+1j$	$u\bar{u} \to W^+W^-Zg$	779	*7 050
$WZZ+1j$	$u\bar{d} \to W^+ZZg$	756	*3 300
$WWW+1j$	$u\bar{d} \to W^+W^-W^+g$	569	*1 800
$ZZZZ$	$u\bar{u} \to ZZZZ$	408	*1 070
$WWWW$	$u\bar{u} \to W^+W^-W^+W^-$	496	*1 350
$t\bar{t}b\bar{b}\,(m_b \neq 0)$	$d\bar{d} \to t\bar{t}b\bar{b}$	275	178
	$gg \to t\bar{t}b\bar{b}$	1 530	5 685
$t\bar{t}+2j$	$gg \to t\bar{t}gg$	4 700	13 827
$Zb\bar{b}+1j\,(m_b \neq 0)$	$dug \to ue^+e^-b\bar{b}$	708	*1 070
$Wb\bar{b}+1j\,(m_b \neq 0)$	$u\bar{d} \to e^+\nu_e b\bar{b}g$	312	67
$Wb\bar{b}+2j\,(m_b \neq 0)$	$u\bar{d} \to e^+\nu_e b\bar{b}s\bar{s}$	648	181
	$u\bar{d} \to e^+\nu_e b\bar{b}d\bar{d}$	1 220	895
	$u\bar{d} \to e^+\nu_e b\bar{b}gg$	3 923	5387
$WWb\bar{b}\,(m_b \neq 0)$	$d\bar{d} \to \nu_e e^+ \bar{\nu}_\mu \mu^- b\bar{b}$	292	115
	$gg \to \nu_e e^+ \bar{\nu}_\mu \mu^- b\bar{b}$	1 068	*5 300
$WWb\bar{b}+1j\,(m_b = 0)$	$u\bar{u} \to \nu_e e^+ \bar{\nu}_\mu \mu^- b\bar{b}g$	3 612	*2 000
$H+3j$ in GF	$gg \to Hggg$	9 325	8 961
$t\bar{t}Z+1j$	$u\bar{u} \to t\bar{t}e^+e^-g$	1408	1 220
	$gg \to t\bar{t}e^+e^-g$	4230	19 560
$t\bar{t}H+1j$	$gg \to t\bar{t}Hg$	1 517	1 505
$H+3j$ in VBF	$u\bar{u} \to Hgu\bar{u}$	432	101
$H+4j$ in VBF	$u\bar{u} \to Hggu\bar{u}$	1 176	669
$H+5j$ in VBF	$u\bar{u} \to Hgggu\bar{u}$	15 036	29 200

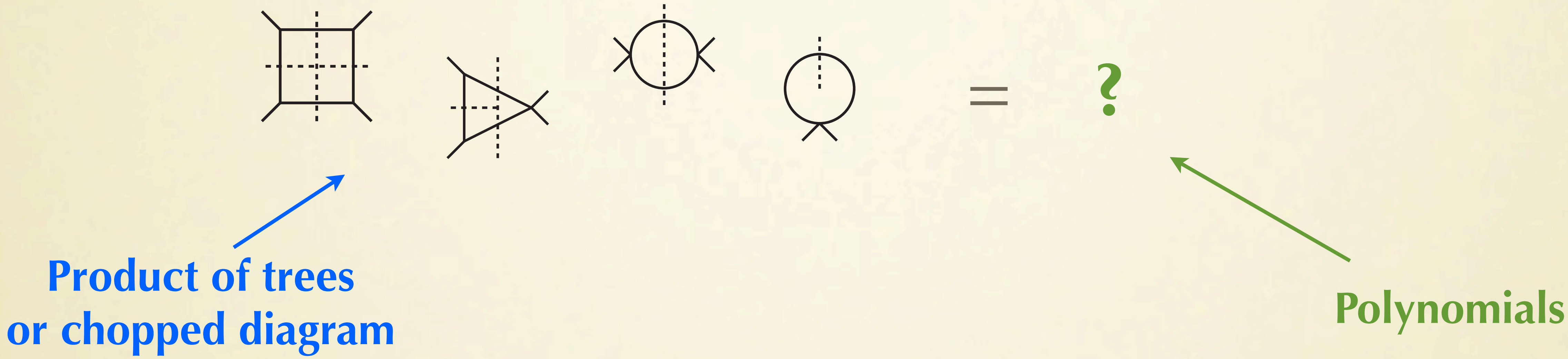
8-particle with internal and external masses

Table 2: A summary of results obtained with GoSam+Ninja. Timings refer to full color- and helicity-summed amplitudes, using an Intel Core i7 CPU @ 3.40GHz, compiled with `ifort`. The timings indicated with an (*) are obtained with an Intel(R) Xeon(R) CPU E5-2650 0 @ 2.00GHz, compiled with `gfortran`.

faster,
higher accuracy,
more stable,
no-problem with
multiple masses

Towards Higher Loop

Problem: what is the form of the residues?



"find the right variables encoding the cut-structure"

variables

- ISP's = Irreducible Scalar Products:
  - q-components which can variate under cut-conditions
  - spurious: vanishing upon integration
  - non-spurious: non-vanishing upon integration $\Rightarrow$ MI's

Ossola & **P.M.** (2011)

Zhang (2012); Badger Frellesvig Zhang (2012)
Mirabella, Ossola, Peraro, & **P.M.** (2012)

- Quantum Field Theory
  - Unitarity-Cuts, Vanishing denominators
  - Cut-residue
  - Amplitudes factorization in tree-amplitudes

Amplitude decomposition

- Algebraic Geometry
  - Polynomial equations, ideals
  - Remainder of polynomial division
  - Polynomials in quotient rings

Multivariate Polynomial division

>> Zhang, Badger

Multivariate Polynomial Division

Zhang (2012); Badger Frellesvig Zhang (2012)
Mirabella, Ossola, Peraro, & **P.M.** (2012)

- **Ideal** $$\mathcal{J}_{i_1\cdots i_n} = \langle D_{i_1}, \cdots, D_{i_n} \rangle \equiv \left\{ \sum_{\kappa=1}^{n} h_\kappa(\mathbf{z}) D_{i_\kappa}(\mathbf{z}) : h_\kappa(\mathbf{z}) \in P[\mathbf{z}] \right\}$$

- **Groebner Basis** $$\mathcal{G}_{i_1\cdots i_n} = \{g_1(\mathbf{z}), \ldots, g_m(\mathbf{z})\}$$

$$\mathcal{J}_{i_1\ldots i_n} = \langle g_1, \ldots, g_m \rangle \equiv \left\{ \sum_{\kappa=1}^{m} \tilde{h}_\kappa(\mathbf{z}) g_\kappa(\mathbf{z}) : \tilde{h}_\kappa(\mathbf{z}) \in P[\mathbf{z}] \right\}$$

n-ple cut-conditions $$D_{i_1} = \ldots = D_{i_n} = 0 \quad \Leftrightarrow \quad g_1 = \ldots = g_m = 0$$

Multivariate Polynomial Division

Zhang (2012); Badger Frellesvig Zhang (2012)
Mirabella, Ossola, Peraro, & **P.M.** (2012)

Ideal $\mathcal{J}_{i_1\cdots i_n} = \langle D_{i_1}, \cdots, D_{i_n} \rangle \equiv \left\{ \sum_{\kappa=1}^{n} h_\kappa(\mathbf{z}) D_{i_\kappa}(\mathbf{z}) : h_\kappa(\mathbf{z}) \in P[\mathbf{z}] \right\}$

Groebner Basis $\mathcal{G}_{i_1\cdots i_n} = \{g_1(\mathbf{z}), \ldots, g_m(\mathbf{z})\}$

$$\mathcal{J}_{i_1\ldots i_n} = \langle g_1, \ldots, g_m \rangle \equiv \left\{ \sum_{\kappa=1}^{m} \tilde{h}_\kappa(\mathbf{z}) g_\kappa(\mathbf{z}) : \tilde{h}_\kappa(\mathbf{z}) \in P[\mathbf{z}] \right\}$$

n-ple cut-conditions $D_{i_1} = \ldots = D_{i_n} = 0 \quad \Leftrightarrow \quad g_1 = \ldots = g_m = 0$

Polynomial Division $\mathcal{N}_{i_1\cdots i_n}(\mathbf{z}) = \Gamma_{i_1\cdots i_n} + \Delta_{i_1\cdots i_n}(\mathbf{z})$,

Remainder ~ Residue $\Delta_{i_1\cdots i_n}(\mathbf{z})$

Quotients $\Gamma_{i_1\cdots i_n} = \sum_{i=1}^{m} \mathcal{Q}_i(\mathbf{z}) g_i(\mathbf{z})$ belongs to the ideal $\mathcal{J}_{i_1\cdots i_n}$,

$$= \sum_{\kappa=1}^{n} \mathcal{N}_{i_1\cdots i_{\kappa-1} i_{\kappa+1}\cdots i_n}(\mathbf{z}) D_{i_\kappa}(\mathbf{z}) \; .$$

Multi-Loop Integrand Recurrence

Mirabella, Ossola, Peraro, & ***P.M.*** (2012)

$$\frac{\mathcal{N}_{i_1\ldots i_n}}{D_{i_1}\cdots D_{i_n}} = \sum_{\kappa=1}^{n} \frac{\mathcal{N}_{i_1\ldots i_{\kappa-1} i_{\kappa+1}\ldots i_n}\; D_{i_\kappa}}{D_{i_1}\cdots D_{i_{\kappa-1}} D_{i_\kappa} D_{i_{\kappa+1}}\cdots D_{i_n}} + \frac{\Delta_{i_1\ldots i_n}}{D_{i_1}\cdots D_{i_n}}$$

Multi-Loop Integrand Recurrence

Mirabella, Ossola, Peraro, & ***P.M.*** (2012)

$$\frac{\mathcal{N}_{i_1\ldots i_n}}{D_{i_1}\cdots D_{i_n}} = \sum_{\kappa=1}^{n} \frac{\mathcal{N}_{i_1\ldots i_{\kappa-1} i_{\kappa+1}\ldots i_n} \, \cancel{D_{i_\kappa}}}{D_{i_1}\cdots D_{i_{\kappa-1}} \cancel{D_{i_\kappa}} D_{i_{\kappa+1}}\cdots D_{i_n}} + \frac{\Delta_{i_1\ldots i_n}}{D_{i_1}\cdots D_{i_n}}$$

remainder = residue

$$\mathcal{I}_{i_1\cdots i_n} = \sum_{\kappa=1}^{k} \mathcal{I}_{i_1\cdots i_{\kappa-1} i_{\kappa+1} i_n} + \frac{\Delta_{i_1\cdots i_n}}{D_{i_1}\cdots D_{i_n}} \, .$$

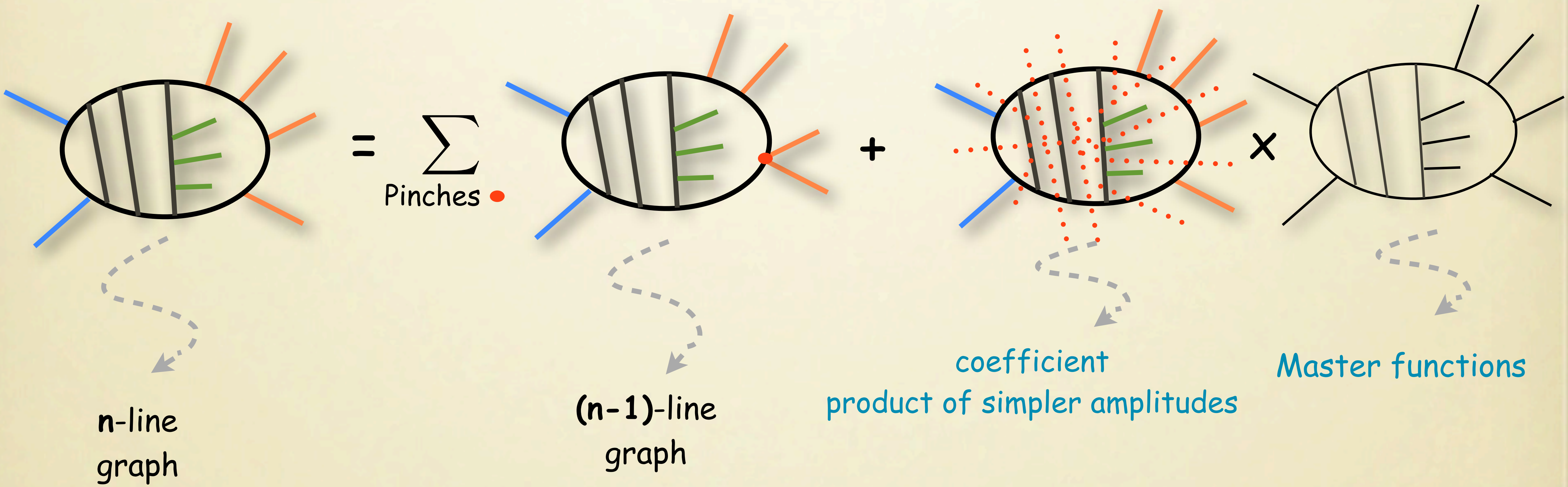
n-denominator integrand

(n-1)-denominator integrand

Multi-Loop Integrand Recurrence

Mirabella, Ossola, Peraro, & ***P.M.*** (2012)

- **ℓ-Loop Recurrence Relation**



- ☑ **all orders** (any number of loops and legs)
- ☑ **any topology** (planar and non-planar)
- ☑ **all kinematics** (massless and massive)
- ☑ **high-power of denominators**

Multi-Loop Integrand Decomposition

Mirabella, Ossola, Peraro, & ***P.M.*** (2012)

Divide & Conquer approach

$$\mathcal{I}_{i_1 \cdots i_n} = \frac{\mathcal{N}_{i_1 \cdots i_n}}{D_{i_1} D_{i_2} \cdots D_{i_n}}$$

$$\mathcal{I}_{i_1 \cdots i_n} = \sum_{1=i_1 << i_{\max}}^{n} \frac{\Delta_{i_1 i_2 \ldots i_{\max}}}{D_{i_1} D_{i_2} \cdots D_{i_{\max}}} + \sum_{1=i_1 << i_{\max}-1}^{n} \frac{\Delta_{i_1 i_2 \ldots i_{\max}-1}}{D_{i_1} D_{i_2} \cdots D_{i_{\max}-1}}$$

$$+ \sum_{1=i_1 << i_{\max}-2}^{n} \frac{\Delta_{i_1 i_2 \ldots i_{\max}-2}}{D_{i_1} D_{i_2} \cdots D_{i_{\max}-2}} + \cdots\cdots + \sum_{1=i_1 < i_2}^{n} \frac{\Delta_{i_1 i_2}}{D_{i_1} D_{i_2}} + \sum_{1=i_1}^{n} \frac{\Delta_{i_1}}{D_{i_1}} + Q_{\emptyset}$$

SINGULAR @ work!

The Maximum-Cut Theorem

Mirabella, Ossola, Peraro, & **P.M.** (2012)

At any loop ℓ, loops we define *maximum cut* as the set of vanishing denominators

$$D_0 = D_1 = \ldots = 0$$

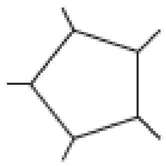
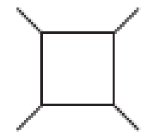
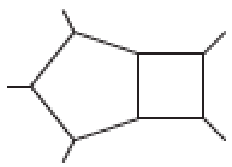
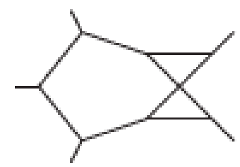
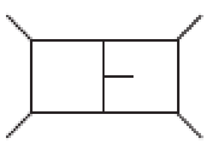
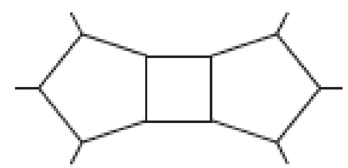
which constrains completely the components of the loop momenta. 0-dimensional

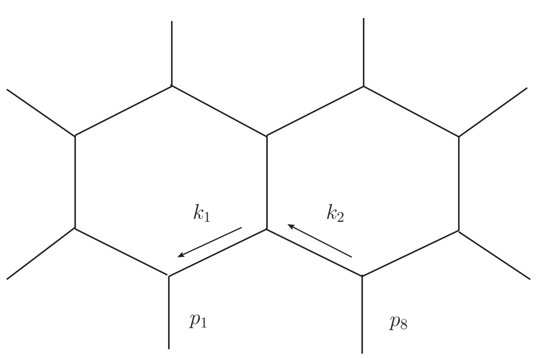
We assume that, in non-exceptional phase-space points, a maximum-cut has a finite number n_s of solutions, each with multiplicity one.

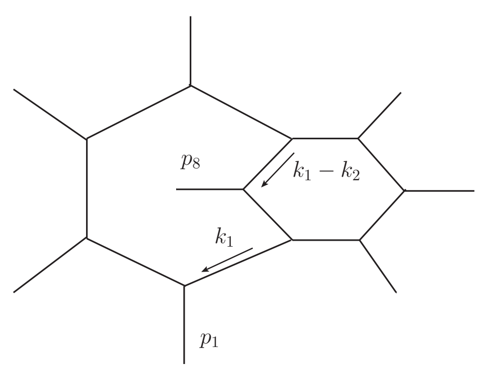
Then,

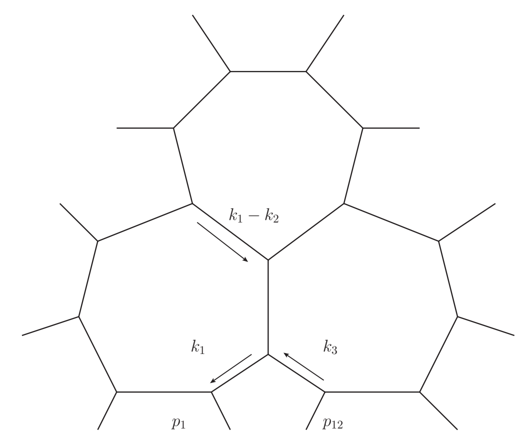
Theorem 4.1 (Maximum cut)**.** *The residue at the maximum-cut is a polynomial parametrised by n_s coefficients, which admits a univariate representation of degree $(n_s - 1)$.*

Examples of Maximum-Cuts

diagram	Δ	n_s	diagram	Δ	n_s
	c_0	1		$c_0 + c_1 z$	2
	$\sum_{i=0}^{3} c_i z^i$	4		$\sum_{i=0}^{3} c_i z^{\,i}$	4
	$\sum_{i=0}^{7} c_i z^{\,i}$	8		$\sum_{i=0}^{7} c_i z^{\,i}$	8







One-Loop Integrand Decomposition $\quad d = 4 - 2\epsilon$

- Choice of 4-dimensional basis for an m-point residue

$$e_1^2 = e_2^2 = 0\,, \qquad e_1 \cdot e_2 = 1\,, \qquad e_3^2 = e_4^2 = \delta_{m4}\,, \qquad e_3 \cdot e_4 = -(1 - \delta_{m4})$$

- Coordinates: $\mathbf{z} = (z_1, z_2, z_3, z_4, z_5) \equiv (x_1, x_2, x_3, x_4, \mu^2)$

$$q^{\mu}_{\text{4-dim}} = -p^{\mu}_{i_1} + x_1\, e_1^{\mu} + x_2\, e_2^{\mu} + x_3\, e_3^{\mu} + x_4\, e_4^{\mu}\,, \qquad q^2 = q^2_{\text{4-dim}} - \mu^2$$

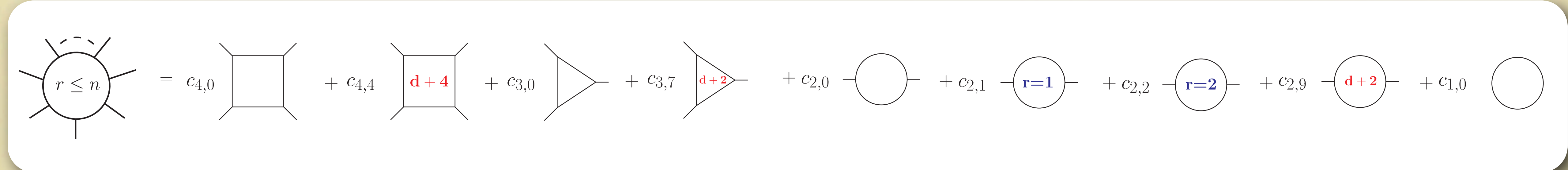
- Generic numerator

$$\mathcal{N}_{i_1\cdots i_m} = \sum_{j_1,\ldots,j_5} \alpha_{\vec{j}}\, z_1^{j_1}\, z_2^{j_2}\, z_3^{j_3}\, z_4^{j_4}\, z_5^{j_5}\,, \qquad (j_1 \ldots j_5) \quad \text{such that} \quad \text{rank}(\mathcal{N}_{i_1\cdots i_m}) \leq m$$

- Residues

reproducing:
Ossola Papadopoulos Pittau
Ellis Giele Kunszt Melnikov

$$\Delta_{i_1 i_2 i_3 i_4 i_5} = c_0$$
$$\Delta_{i_1 i_2 i_3 i_4} = c_0 + c_1 x_4 + \mu^2 (c_2 + c_3 x_4 + \mu^2 c_4)$$
$$\Delta_{i_1 i_2 i_3} = c_0 + c_1 x_3 + c_2 x_3^2 + c_3 x_3^3 + c_4 x_4 + c_5 x_4^2 + c_6 x_4^3 + \mu^2 (c_7 + c_8 x_3 + c_9 x_4)$$
$$\Delta_{i_1 i_2} = c_0 + c_1 x_2 + c_2 x_3 + c_3 x_4 + c_4 x_2^2 + c_5 x_3^2 + c_6 x_4^2 + c_7 x_2 x_3 + c_9 x_2 x_4 + c_9 \mu^2$$
$$\Delta_{i_1} = c_0 + c_1 x_1 + c_2 x_2 + c_3 x_3 + c_4 x_4$$



Longitudinal and Transverse Space

- **Dimensional Regularization**

$$d = 4 - 2\epsilon$$

- **if n-legs < 5**

$$d = d_{//} + d_{\perp}$$

Longitudinal space spanned by the (independent) legs

Transverse Space

☑ Denominators **do not depend on** "the angular variables" of the Transverse Space $\Omega_{\perp}$

☑ Numerators **depend on** "all" loop variables

Integrating over Transverse Angles

Peraro Primo ***P.M.*** (to appear)

- **Spherical Coordinates**

@ 1-loop

$$I_1 = \int d^n \boldsymbol{\lambda}\, \mathcal{I}_1(\boldsymbol{\lambda}), \qquad n = d_\perp$$

$$\boldsymbol{\lambda} = \sum_{i=1}^{n} a_i \mathbf{v}_i, \qquad \mathbf{v}_i \cdot \mathbf{v}_j = \delta_{ij}. \qquad \mathcal{I}_1(\boldsymbol{\lambda}) \equiv \mathcal{I}_1(\lambda^2, \{a_1, a_2, \ldots, a_k\}).$$

$$\begin{cases} a_1 & = \lambda \cos\theta_1 \\ a_2 & = \lambda \sin\theta_1 \cos\theta_2 \\ & \cdots \\ a_k & = \lambda \cos\theta_k \prod_{i=1}^{k-1} \sin\theta_i. \end{cases}$$

$$I_1 = \frac{\pi^{\frac{n-k}{2}}}{\Gamma\left(\frac{n-k}{2}\right)} \int_0^\infty d\lambda^2 (\lambda^2)^{\frac{n-2}{2}} \prod_{i=1}^{k} \int_{-1}^{1} d\cos\theta_i (\sin\theta_i)^{n-i-2} \mathcal{I}_1(\lambda^2, \{\cos\theta_i, \sin\theta_i\})$$

Integrating over Transverse Angles

Peraro Primo ***P.M.*** (to appear)

- **Spherical Coordinates**

@ 2-loop

$$I_2 = \int d^n \boldsymbol{\lambda}_1 d^n \boldsymbol{\lambda}_2 \mathcal{I}_2(\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2), \qquad n = d_\perp$$

$$\boldsymbol{\lambda}_1 = \sum_{i=1}^{n} a_i \mathbf{v}_i, \qquad \boldsymbol{\lambda}_2 = \sum_{i=1}^{n} b_i \mathbf{v}_i. \qquad \lambda_{ij} = \boldsymbol{\lambda}_i \cdot \boldsymbol{\lambda}_j \qquad \mathcal{I}_2(\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2) = \mathcal{I}_2(\lambda_{ij}, \{a_1, a_2, \ldots, a_k\}, \{b_1, b_2, \ldots, b_k\}).$$

$$\cos\theta_{12} = \frac{\lambda_{12}}{\sqrt{\lambda_{11}\lambda_{22}}},$$

$$\begin{cases} a_1 = & \sqrt{\lambda_{11}} \cos\theta_{11} \\ & \ldots \\ a_k = & \sqrt{\lambda_{11}} \cos\theta_{k1} \prod_{i=1}^{k-1} \sin\theta_{i1} \end{cases}$$

$$\begin{cases} b_1 = & \sqrt{\lambda_{22}} \left(\cos\theta_{12}\cos\theta_{11} + \cos\theta_{22}\sin\theta_{11}\sin\theta_{12}\right) \\ & \ldots \\ b_i = & \sqrt{\lambda_{22}} \big[\cos\theta_{12}\cos\theta_{i1} \prod_{j=1}^{i-1}\sin\theta_{j1} + \cos\theta_{i+1\,2}\sin\theta_{i1}\prod_{j=1}^{i}\sin\theta_{j2} \\ & - \cos\theta_{i1}\sum_{k=2}^{i}\cos\theta_{k2}\cos\theta_{k-1\,1}\prod_{j=1}^{k-1}\sin\theta_{j2}\big(\delta_{ik} + (1-\delta_{ik})\prod_{l=1}^{i-k}\sin\theta_{k+l-1\,1}\big)\big]. \end{cases}$$

$$I_2 = \frac{(2\pi)^{n-k-1}}{2\Gamma(n-k-1)} \int_0^\infty d\lambda_{11} (\lambda_{11})^{\frac{n-2}{2}} \int_0^\infty d\lambda_{22} (\lambda_{22})^{\frac{n-2}{2}} \int_{-1}^{1} d\cos\theta_{12} (\sin\theta_{12})^{n-3} \times$$

$$\int_{-1}^{1} \prod_{i=1}^{k} d\cos\theta_{i1} d\cos\theta_{i+1\,2} (\sin\theta_{i1})^{n-i-2} (\sin\theta_{i+1\,2})^{n-i-3} \mathcal{I}_2(\lambda_{11}, \lambda_{22}, \{\cos\theta_{i1,2}, \sin\theta_{i1,2}\})$$

@ higher-loop... as well

Gegenbauer Polynomials

Orthogonal polynomials

orthogonal polynomials over the interval $[-1, 1]$

weight function $\omega_\alpha(x) = (1-x^2)^{\alpha-\frac{1}{2}}$

generating function $\dfrac{1}{(1-2xt+t^2)^\alpha} = \sum_{n=1}^{\infty} C_n^{(\alpha)}(x) t^n.$

$$
\begin{aligned}
C_0^{(\alpha)}(x) &= 1, \\
C_1^{(\alpha)}(x) &= 2\alpha x, \\
C_2^{(\alpha)}(x) &= -\alpha + 2\alpha(1+\alpha)x^2,
\end{aligned}
$$

$$
\begin{aligned}
x &= \frac{1}{2\alpha} C_0^{(\alpha)}(x) C_1^{(\alpha)}(x), \\
x^2 &= \frac{1}{4\alpha^2} [C_1^{(\alpha)}(x)]^2, \\
x^3 &= \frac{1}{4\alpha^2(1+\alpha)} C_1^{(\alpha)}(x)[\alpha C_0^{(\alpha)}(x) + C_2^{(\alpha)}(x)], \\
x^4 &= \frac{1}{4\alpha^2(1+\alpha)^2} [\alpha C_0^{(\alpha)}(x) + C_2^{(\alpha)}(x)]^2, \\
&\dots
\end{aligned}
$$

Orthogonality condition

$$
\int_{-1}^{1} d\cos\theta (\sin\theta)^{2\alpha-1} C_n^{(\alpha)}(\cos\theta) C_m^{(\alpha)}(\cos\theta) = \delta_{mn} \frac{2^{1-2\alpha}\pi\Gamma(n+2\alpha)}{n!(n+\alpha)\Gamma^2(\alpha)}.
$$

☑ **Integration over Transverse Angles: trivialized @ all-loop!**

Peraro Primo ***P.M.***

One-Loop Integrals

$$d = 4 - 2\epsilon$$

$$I_n^d[\mathcal{N}] = \int \frac{d^d q}{\pi^{d/2}} \frac{\mathcal{N}(q)}{\prod_{i=0}^{n-1} \mathcal{D}_i}, \qquad \mathcal{D}_i = \Big(q + \sum_{j=0}^{i} p_j\Big)^2 + m_i^2, \qquad p_0 = 0,$$

- loop momentum parametrization

$$q^\alpha = q_{[4]}^\alpha + \mu^\alpha, \qquad q_{[4]}^\alpha = \sum_{i=1}^{4} x_i e_i^\alpha, \qquad q^2 = q_{[4]}^2 + \mu^2.$$

- Integration variables

$$I_n^d[\mathcal{N}] = \frac{\mathcal{K}}{\pi^2 \Gamma\left(\frac{d-4}{2}\right)} \int_{-\infty}^{\infty} \prod_{i=1}^{4} dx_i \int_0^\infty d\mu^2 (\mu^2)^{\frac{d-6}{2}} \frac{\mathcal{N}(x_i, \mu^2)}{\prod_{i=0}^{n-1} \mathcal{D}_i},$$

$$\mathcal{D}_i = \Big(q_{[4]} + \sum_{j=0}^{i} p_j\Big)^2 + \mu^2 + m_i^2, \qquad \mathcal{K} = \sqrt{\det\left(\frac{\partial q_{[4]}^\mu}{\partial x_i} \frac{\partial q_{[4]\,\mu}}{\partial x_j}\right)}.$$

One-Loop Integrals $d = d_{//} + d_{\perp}$

- loop momentum parametrization

$$q^\alpha = q^\alpha_{[k]} + \lambda^\alpha, \qquad q^\alpha_{[k]} = \sum_{j=1}^{k} x_j e^\alpha_j, \qquad q^2 = q^2_{[k]} + \lambda^2,$$

k-dimensional the space spanned by the external momenta

$$\lambda^\alpha = \sum_{j=k+1}^{4} x_j e^\alpha_j + \mu^\alpha, \qquad \lambda^2 = \sum_{j=k+1}^{4} x_j^2 + \mu^2,$$

$(d-k)$-dimensional orthogonal subspace.

$$I_n^d[\mathcal{N}] = \frac{1}{\pi^2 \Gamma\left(\frac{d-4}{2}\right)} \int d^k q_{[k]} \int_0^\infty d\lambda^2 (\lambda^2)^{\frac{d-k-2}{2}} \prod_{i=1}^{4-k} \int_{-1}^{1} d\cos\theta_i (\sin\theta_i)^{d-k-i-2} \frac{\mathcal{N}(q)}{\prod_{i=0}^{n-1} \mathcal{D}_i}.$$

$$\mathcal{N}(q) \equiv \mathcal{N}(q^\alpha_{[k]}, \lambda^2, \{x_{k+1}, \ldots, x_4\}). \qquad \mathcal{D}_i = \left(q_{[k]} + \sum_{j=0}^{i} p_j\right)^2 + \lambda^2 + m_i^2.$$

☑ **Denominators do not depend on** "the angular variables" of the **Transverse Space** $\Omega_\perp$

☑ **Integration over** $\Omega_\perp$ **: *Gegenbauer orthogonality condition***

Spurious integrals vanish automatically!

Four-point integrals

$$I_4^d[\mathcal{N}] = \int \frac{d^3 q_{[3]}}{\pi^{d/2}} \int d^{d-3}\lambda \frac{\mathcal{N}(q_{[3]}, \lambda^2, x_4)}{\mathcal{D}_0\mathcal{D}_1\mathcal{D}_2\mathcal{D}_3}.$$

$$x_4 = \lambda \cos\theta_1$$

$$I_4^d[\mathcal{N}] = \frac{1}{\pi^2\Gamma\left(\frac{d-4}{2}\right)} \int d^3 q_{[3]} \int_0^\infty d\lambda^2 (\lambda^2)^{\frac{d-5}{2}} \int_{-1}^{1} d\cos\theta_1 (\sin\theta_1)^{d-6} \frac{\mathcal{N}(q_{[3]}, \lambda^2, \cos\theta_1)}{\mathcal{D}_0\mathcal{D}_1\mathcal{D}_2\mathcal{D}_3}.$$

Examples

$$\cos^2\theta_1 = \frac{1}{(d-5)^2}\left[C_1^{(\frac{d-5}{2})}(\cos\theta_1)\right]^2,$$

$$\cos^4\theta_1 = \frac{1}{(d-3)^2}\left[C_0^{(\frac{d-5}{2})}(\cos\theta_1) + \frac{4}{(d-5)^2} C_2^{\frac{d-5}{2}}(\cos\theta_1)\right]^2$$

$$I_4^d[x_4^2] = \frac{1}{d-3} I_4^d[\lambda^2] = \frac{1}{2} I_4^{d+2}[1],$$

$$I_4^d[x_4^4] = \frac{3}{(d-3)(d-1)} I_4^d[\lambda^4] = \frac{3}{4} I_4^{d+4}[1].$$

Gegenbauer integration produces powers of $\lambda_{ij} = \boldsymbol{\lambda}_i \cdot \boldsymbol{\lambda}_j$

Three-point integrals

$$\begin{cases} x_3 = \lambda\cos\theta_1 \\ x_4 = \lambda\sin\theta_1\cos\theta_2 \end{cases}$$

$$I_3^d[\mathcal{N}\,] = \frac{1}{\pi^2\Gamma\left(\frac{d-4}{2}\right)}\int d^2q_{[2]}\int_0^\infty d\lambda^2(\lambda^2)^{\frac{d-4}{2}}\int_{-1}^1 d\cos\theta_1(\sin\theta_1)^{d-5}\times\int_{-1}^1 d\cos\theta_2(\sin\theta_2)^{d-6}\frac{\mathcal{N}(q_{[2]},\lambda^2,\{\cos\theta_1,\sin\theta_1,\cos\theta_2\})}{\mathcal{D}_0\mathcal{D}_1\mathcal{D}_2}.$$

Two-point integrals

$$\begin{cases} x_2 = \lambda\cos\theta_1 \\ x_3 = \lambda\sin\theta_1\cos\theta_2, \\ x_4 = \lambda\sin\theta_1\sin\theta_2\cos\theta_3 \end{cases}$$

$$I_2^d[\mathcal{N}\,] = \frac{1}{\pi^2\Gamma\left(\frac{d-4}{2}\right)}\int dq_{[1]}\int_0^\infty d\lambda^2(\lambda^2)^{\frac{d-3}{2}}\int_{-1}^1 d\cos\theta_1(\sin\theta_1)^{d-4}\times\int_{-1}^1 d\cos\theta_2(\sin\theta_2)^{d-5}\int_{-1}^1 d\cos\theta_3(\sin\theta_3)^{d-6}\times \frac{\mathcal{N}(q_{[1]},\lambda^2,\cos\theta_1,\sin\theta_1,\cos\theta_2,\sin\theta_2,\cos\theta_3)}{\mathcal{D}_0\mathcal{D}_1},$$

$$I_2^d[\mathcal{N}]\big|_{p^2=0} = \frac{1}{\pi^2\Gamma\left(\frac{d-4}{2}\right)}\int d^2q_{[2]}\int_0^\infty d\lambda^2(\lambda^2)^{\frac{d-4}{2}}\int_{-1}^1 d\cos\theta_1(\sin\theta_1)^{d-5}\times\int_{-1}^1 d\cos\theta_2(\sin\theta_2)^{d-6}\frac{\mathcal{N}(q_{[2]},\lambda^2,\cos\theta_1,\sin\theta_1,\cos\theta_2)}{\mathcal{D}_0\mathcal{D}_1};$$

$$\begin{cases} x_3 = \lambda\cos\theta_1 \\ x_4 = \lambda\sin\theta_1\cos\theta_2 \end{cases}$$

One-point integrals

$$\begin{cases} x_1 = \lambda\cos\theta_1, \\ x_2 = \lambda\sin\theta_1\cos\theta_2, \\ x_3 = \lambda\sin\theta_1\sin\theta_2\cos\theta_3 \\ x_4 = \lambda\sin\theta_1\sin\theta_2\sin\theta_3\cos\theta_4 \end{cases}$$

$$I_1^d[\mathcal{N}\,] = \frac{1}{\pi^2\Gamma\left(\frac{d-4}{2}\right)}\int_0^\infty d\lambda^2(\lambda^2)^{\frac{d-2}{2}}\int_{-1}^1 d\cos\theta_1(\sin\theta_1)^{d-3}\int_{-1}^1 d\cos\theta_1(\sin\theta_1)^{d-4}\times\int_{-1}^1 d\cos\theta_2(\sin\theta_2)^{d-5}\times\int_{-1}^1 d\cos\theta_3(\sin\theta_3)^{d-6}\times \frac{\mathcal{N}(q_{[1]},\lambda^2,\cos\theta_1,\sin\theta_1,\cos\theta_2,\sin\theta_2,\cos\theta_3,\sin\theta_3,\cos\theta_4)}{\mathcal{D}_0}$$

One-Loop Integrand Decomposition

$d = d_{//} + d_{\perp}$

- **Adaptive Unitarity**

$$q^\alpha = q^\alpha_{[k]} + \lambda^\alpha, \qquad \mathcal{D}_i = \left(q_{[k]} + \sum_{j=0}^{i} p_j\right)^2 + \lambda^2 + m_i^2.$$

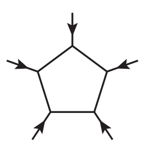
$$q^\alpha_{[k]} = \sum_{j=1}^{k} x_j e^\alpha_j, \qquad \lambda^2$$

reducible

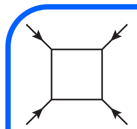
Cutting in different dimensions according to the # of legs

1-loop :: always MAXIMUM CUTS

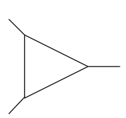
New residue parametrization



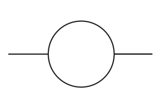
$$\Delta_{i_0 \cdots i_4} = c_0.$$



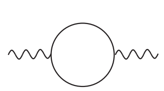
$$\Delta_{i_0 \cdots i_3} = c_0 + c_1 x_4 + c_2 x_4^2 + c_3 x_4^3 + c_4 x_4^4,$$



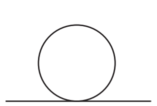
$$\Delta_{i_0 i_1 i_2} = c_0 + c_1 x_3 + c_2 x_4 + c_3 x_3^2 + c_4 x_3 x_4 + c_5 x_4^2 + c_6 x_3^3 + c_7 x_3^2 x_4 + c_8 x_3 x_4^2 + c_9 x_4^3.$$



$$\Delta_{i_0 i_1} = c_0 + c_1 x_2 + c_2 x_3 + c_3 x_4 + c_4 x_2 x_3 + c_5 x_2 x_4 + c_6 x_3 x_4 + c_7 x_2^2 + c_8 x_3^2 + c_9 x_4^2.$$



$$\Delta_{i_0 i_1}|_{p^2=0} = c_0 + c_1 x_1 + c_2 x_3 + c_3 x_4 + c_4 x_1 x_3 + c_5 x_1 x_4 + c_6 x_3 x_4 + c_7 x_1^2 + c_8 x_3^2 + c_9 x_4^2.$$



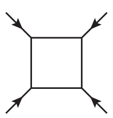
$$\Delta_{i_0} = c_0 + \sum_{i=1}^{4} c_i x_i$$

One-Loop Integrand Decomposition

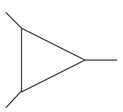
$$d = d_{//} + d_{\perp}$$

Adaptive Unitarity

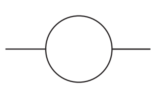
Integration of the Residues over Transverse Angles



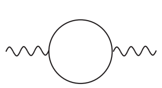
$$\int \frac{d^d q}{\pi^{d/2}} \frac{\Delta_{i_0 i_1 i_2 i_3}}{\mathcal{D}_{i_0}\mathcal{D}_{i_1}\mathcal{D}_{i_2}\mathcal{D}_{i_3}} = c_0 I_4^d[1] + \frac{1}{(d-3)} c_2 I_4^d[\lambda^2] + \frac{3}{(d-3)(d-1)} c_4 I_4^d[\lambda^4] = c_0 I_4^d[1] + \frac{1}{2} c_2 I_4^{d+2}[1] + \frac{3}{4} c_4 I_4^{d+4}[1].$$



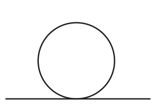
$$\int \frac{d^d q}{\pi^{d/2}} \frac{\Delta_{i_0 i_1 i_2}}{\mathcal{D}_{i_0}\mathcal{D}_{i_1}\mathcal{D}_{i_2}} = c_0 I_3^d[1] + \frac{1}{(d-3)} (c_3 + c_5) I_3^d[\lambda^2] = c_0 I_3^d[1] + \frac{1}{2}(c_3 + c_5) I_3^{d+2}[1].$$



$$\int \frac{d^d q}{\pi^{d/2}} \frac{\Delta_{i_0 i_1}}{\mathcal{D}_{i_0}\mathcal{D}_{i_1}} = c_0 I_2^d[1] + \frac{1}{(d-3)} (c_7 + c_8 + c_9) I_2^d[\lambda^2] = c_0 I_2^d[1] + \frac{1}{2}(c_7 + c_8 + c_9) I_2^{d+2}[1].$$



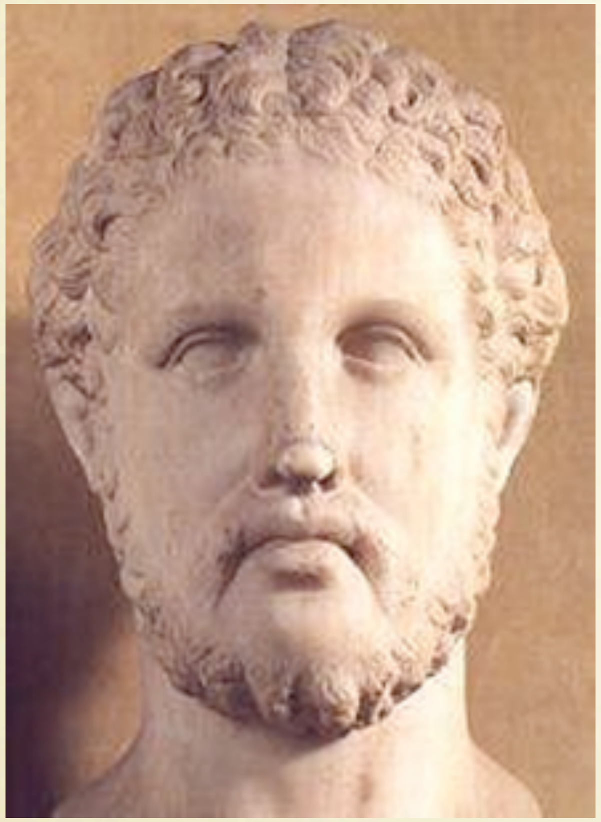
$$\int \frac{d^d q}{\pi^{d/2}} \frac{\Delta_{i_0 i_1}}{\mathcal{D}_{i_0}\mathcal{D}_{i_1}}\Bigg|_{p^2=0} = c_0 I_2^d[1] + c_1 I_2^d[x_1] + c_7 I_2^d[x_1^2] + \frac{1}{(d-3)} (c_8 + c_9) I_3^d[\lambda^2] = c_0 I_2^d[1] + c_1 I_2^d[x_1] + c_7 I_2^d[x_1^2] + \frac{1}{2}(c_8 + c_9) I_2^{d+2}[1].$$



$$\int \frac{d^d q}{\pi^{d/2}} \frac{\Delta_{i_0}}{\mathcal{D}_{i_0}} = c_0 I_1^d[1].$$

λ^2

reducible



Divide et Impera

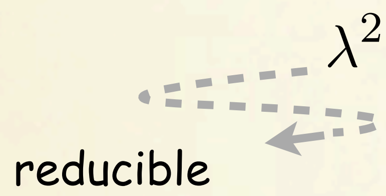
Philip II of Macedon

Divide et Integra...
...et Divide

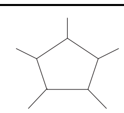
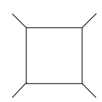
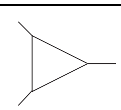
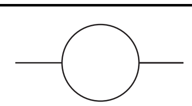
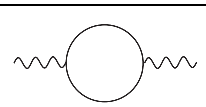
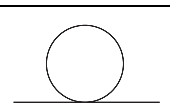
Divide-et-Integra-et-Divide

Peraro Primo ***P.M.***

Additional Polynomial Division



divide | integra | divide

Topology		$\Delta_{i_0 \cdots i_n}$	$\Delta^{\text{int}}_{i_0 \cdots i_n}$	$\Delta'_{i_0 \cdots i_n}$
$\mathcal{I}_{01234}$		1 $\{1\}$	– –	– –
$\mathcal{I}_{0123}$		5 $\{1, x_4, x_4^2, x_4^3, x_4^4\}$	3 $\{1, \lambda^2, \lambda^4\}$	1 $\{1\}$
$\mathcal{I}_{012}$		10 $\{1, x_3, x_4, x_3^2, x_3 x_4, x_4^2, x_3^3, x_3^2 x_4, x_3 x_4^2, x_4^3\}$	2 $\{1, \lambda^2\}$	1 $\{1\}$
$\mathcal{I}_{02}$		10 $\{1, x_2, x_3, x_4, x_2^2, x_2 x_3, x_2 x_4, x_3^2, x_3 x_4, x_4^2\}$	2 $\{1, \lambda^2\}$	1 $\{1\}$
$\mathcal{I}_{01}$		10 $\{1, x_1, x_3, x_4, x_1^2, x_1 x_3, x_1 x_4, x_3^2, x_3 x_4, x_4^2\}$	4 $\{1, x_1, x_1^2, \lambda^2\}$	3 $\{1, x_1, x_1^2\}$
$\mathcal{I}_{0}$		5 $\{1, x_1, x_2, x_3, x_4\}$	1 $\{1\}$	– –

- ☑ minimal number of irreducible non-spurious monomials (irr. scal. prod.s)!
- ☑ Second polynomial division <==> Dimensional Recurrence @ integrand level

Two-Loop Integrals

$d = 4 - 2\epsilon$

$$I_n^d[\mathcal{N}] = \int \frac{d^d q_1 d^d q_2}{\pi^d} \frac{\mathcal{N}(q_1, q_2)}{\prod_i \mathcal{D}_i}, \qquad q_1^\alpha = q_{1[4]}^\alpha + \mu_1^\alpha, \quad q_2^\alpha = q_{2[4]}^\alpha + \mu_2^\alpha, \qquad \mu_i \cdot \mu_j = \mu_{ij}, \quad q_i \cdot q_j = q_{i[4]} \cdot q_{j[4]} + \mu_{ij},$$

- loop momentum parametrization

$$q_{1[4]}^\alpha = \sum_{i=1}^4 x_i e_i^\alpha, \qquad q_{2[4]}^\alpha = \sum_{i=1}^4 y_i f_i^\alpha,$$

$$I_n^d[\mathcal{N}] = \frac{2^{d-6}\mathcal{K}_1\mathcal{K}_2}{\pi^5\Gamma(d-5)} \int \prod_{i=1}^4 dx_i dy_i \int_0^\infty d\mu_{11} \int_0^\infty d\mu_{22} \int_{-\sqrt{\mu_{11}\mu_{22}}}^{\sqrt{\mu_{11}\mu_{22}}} d\mu_{12} (\mu_{11}\mu_{22} - \mu_{12}^2)^{\frac{d-6}{2}} \times \frac{\mathcal{N}(x_j, y_i, \mu_{ij})}{\prod_i \mathcal{D}_i},$$

Two-Loop Integrals

$$d = d_{//} + d_{\perp}$$

- loop momentum parametrization

$$q_1^\alpha = q_{1[k]}^\alpha + \lambda_1^\alpha, \qquad q_2^\alpha = q_{2[k]}^\alpha + \lambda_2^\alpha, \qquad k \le 3,$$

$$q_{1[k]}^\alpha = \sum_{j=1}^{k} x_j e_j^\alpha, \qquad q_{2[k]}^\alpha = \sum_{j=1}^{k} y_j e_j^\alpha, \qquad k\text{-dimensional space spanned by the external kinematics}$$

$$\lambda_1^\alpha = \sum_{j=k+1}^{4} x_j e_j^\alpha + \mu_1^\alpha, \qquad \lambda_2^\alpha = \sum_{j=k+1}^{4} y_j e_j^\alpha + \mu_2^\alpha \qquad (d-k)\text{-dimensional orthogonal subspaces,}$$

$$\begin{cases} x_{k+1} = & \sqrt{\lambda_{11}} \cos\theta_{11} \\ & \dots \\ x_4 = & \sqrt{\lambda_{11}} \cos\theta_{4-k} \prod_{i=1}^{4-k} \sin\theta_{i1} \end{cases}$$

$$\begin{cases} y_{k+1} = & \sqrt{\lambda_{22}} \left(\cos\theta_{12}\cos\theta_{11} + \cos\theta_{22}\sin\theta_{11}\sin\theta_{12}\right) \\ & \dots \\ y_4 = & \sqrt{\lambda_{22}} \big[\cos\theta_{12}\cos\theta_{4-k\,1} \prod_{j=1}^{4-k-1}\sin\theta_{j1} + \cos\theta_{5-k\,2}\sin\theta_{4-k\,1}\prod_{j=1}^{4-k}\sin\theta_{j2} \\ & -\cos\theta_{4-k\,1}\sum_{l=2}^{4-k}\cos\theta_{l\,2}\cos\theta_{l-1\,1}\prod_{j=1}^{l-1}\sin\theta_{j2}\big(\delta_{4-k\,l} + (1-\delta_{k-4\,l})\prod_{m=1}^{4-k-l}\sin\theta_{l+m-1\,1}\big)\big], \end{cases}$$

$$\cos\theta_{12} = \frac{\lambda_{12}}{\sqrt{\lambda_{11}\lambda_{22}}},$$

$$I_n^d[\mathcal{N}] = \frac{2^{d-6}}{\pi^5 \Gamma(n-k-1)} \int d^k q_{1[k]} d^k q_{2[k]} \int_0^\infty d\lambda_{11} (\lambda_{11})^{\frac{d-k-2}{2}} \int_0^\infty d\lambda_{22} (\lambda_{22})^{\frac{d-k-2}{2}} \times$$
$$\int_{-1}^{1} d\cos\theta_{12} (\sin\theta_{12})^{d-k-3} \int_{-1}^{1} \prod_{i=1}^{4-k} d\cos\theta_{i1} d\cos\theta_{i+1\,2} (\sin\theta_{i1})^{d-k-i-2} (\sin\theta_{i+1\,2})^{d-k-i-3} \times \frac{\mathcal{N}(q_1,q_2)}{\prod_i \mathcal{D}_i}.$$

- ☑ **Denominators do not depend on** “the angular variables” of the **Transverse Space** $\Omega_\perp$
- ☑ **Numerators depend on** “all” loop variables
- ☑ **Integration over** $\Omega_\perp$ **: *Gegenbauer orthogonality condition***

  Spurious integrals vanish automatically @ all-loop!

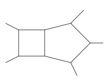
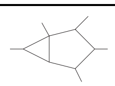
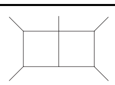
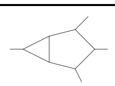
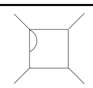
Two-Loop Integrand Decomposition

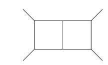
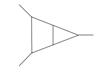
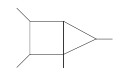
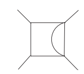
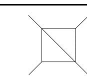
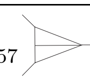
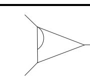
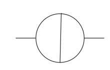
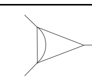
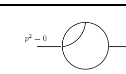
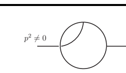
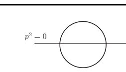
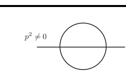
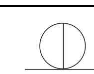
Peraro Primo & ***P.M.***

Divide-et-Integra-et-Divide

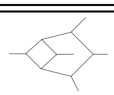
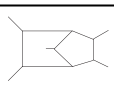
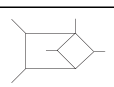
λ_{ij}

reducible

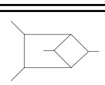
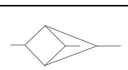
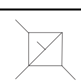
Topology		Δ	$\Delta^{\rm int}$	Δ'
$\mathcal{I}_{12345678}$		60 (+16) $\{1, x_3, x_4, y_4\}$	– –	– –
$\mathcal{I}_{1245678}$		85 (+9) $\{1, x_1, x_3, x_4, y_4\}$	– –	– –
$\mathcal{I}_{1235678}$		145 (+15) $\{1, x_3, x_4, y_3, y_4\}$	– –	– –
$\mathcal{I}_{1345679}$		94 $\{1, x_2, x_3, x_4, y_4\}$	53 $\{1, x_2, x_3, \lambda_{11}, \lambda_{22}, \lambda_{12}\}$	7 (+3) $\{1, x_2, x_3\}$
$\mathcal{I}_{345678}$		66 $\{1, x_1, x_2, x_3, x_4, y_4\}$	35 $\{1, x_1, x_2, x_3, \lambda_{11}, \lambda_{22}, \lambda_{12}\}$	9 (+1) $\{1, x_1, x_2, x_3\}$

Topology		Δ	$\Delta^{\rm int}$	Δ'
$\mathcal{I}_{1234567}$		160 $\{1, x_3, y_4, y_3, y_4\}$	93 $\{1, x_3, y_3, \lambda_{11}, \lambda_{22}, \lambda_{12}\}$	16 (+6) $\{1, x_3, y_3\}$
$\mathcal{I}_{123467}$		180 $\{1, x_3, x_4, y_2, y_3, y_4\}$	22 $\{1, y_2, \lambda_{11}, \lambda_{22}, \lambda_{12}\}$	2 (+2) $\{1, y_2\}$
$\mathcal{I}_{123457}$		180 $\{1, x_3, x_4, y_2, y_3, y_4\}$	101 $\{1, x_3, y_2, y_3, \lambda_{11}, \lambda_{22}, \lambda_{12}\}$	35 (+4) $\{1, x_3, y_2, y_3\}$
$\mathcal{I}_{12357}$		115 $\{1, x_3, x_4, y_1, y_2, y_3, y_4\}$	66 $\{1, x_3, y_1, y_2, y_3, \lambda_{11}, \lambda_{22}, \lambda_{12}\}$	34 (+1) $\{1, x_3, y_1, y_2, y_3\}$
$\mathcal{I}_{12457}$		180 $\{1, x_1, x_3, x_4, y_2, y_3, y_4\}$	103 $\{1, x_1, x_3, y_2, y_3, \lambda_{11}, \lambda_{22}, \lambda_{12}\}$	59 (+1) $\{1, x_1, x_3, y_2, y_3\}$
cut$\mathcal{I}_{23457}$		180 $\{1, x_2, x_3, x_4, y_1, y_3, y_4\}$	33 $\{1, x_2, y_1, \lambda_{11}, \lambda_{22}, \lambda_{12}\}$	12 (+1) $\{1, x_2, y_1\}$
$\mathcal{I}_{12367}$		115 $\{1, x_3, x_4, y_1, y_2, y_3, y_4\}$	20 $\{1, y_1, y_2\lambda_{11}, \lambda_{22}, \lambda_{12}\}$	5 (+1) $\{1, y_1, y_2\}$
$\mathcal{I}_{13467}$		180 $\{1, x_2, x_3, x_4, y_2, y_3, y_4\}$	8 $\{1, \lambda_{11}, \lambda_{22}, \lambda_{12}\}$	1 $\{1\}$
$\mathcal{I}_{2467}$		100 $\{1, x_1, x_2, x_3, x_4, y_2, y_3, y_4\}$	26 $\{1, x_1, x_2, y_2, \lambda_{11}, \lambda_{22}, \lambda_{12}\}$	16 $\{x_1, x_2, y_2\}$
$\mathcal{I}_{1567}$		100 $\{1, x_1, x_2, x_3, x_4, y_1, y_3, y_4\}$	26 $\{1, x_1, x_2, y_1, \lambda_{11}, \lambda_{22}, \lambda_{12}\}$	16 $\{1, x_1, x_2, y_1\}$
$\mathcal{I}_{1467}$		100 $\{1, x_1, x_2, x_3, x_4, y_2, y_3, y_4\}$	8 $\{1, x_1, \lambda_{11}, \lambda_{22}, \lambda_{12}\}$	2 (+1) $\{1, x_1\}$
$\mathcal{I}_{157}$		45 $\{1, x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4\}$	18 $\{1, x_1, x_2, y_1, y_2, \lambda_{11}, \lambda_{22}, \lambda_{12}\}$	15 $\{1, x_1, y_2, x_2, y_2\}$
$\mathcal{I}_{147}$		45 $\{1, x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4\}$	9 $\{1, x_1, y_1, \lambda_{11}, \lambda_{22}, \lambda_{12}\}$	6 $\{1, x_1, y_1\}$
$\mathcal{I}_{167}$		35 $\{1, x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4\}$	4 $\{1, \lambda_{11}, \lambda_{22}, \lambda_{12}\}$	1 $\{1\}$

Planar

Topology		Δ
$\mathcal{I}^{\rm A}_{12345678}$		64 (+16) $\{1, x_3, x_4, y_4\}$
$\mathcal{I}^{\rm B}_{12345678}$		96 (+20) $\{1, x_3, y_3, y_4\}$
$\mathcal{I}^{\rm A}_{1234578}$		170 (+15) $\{1, x_3, x_4, y_3, y_4\}$

Non-Planar

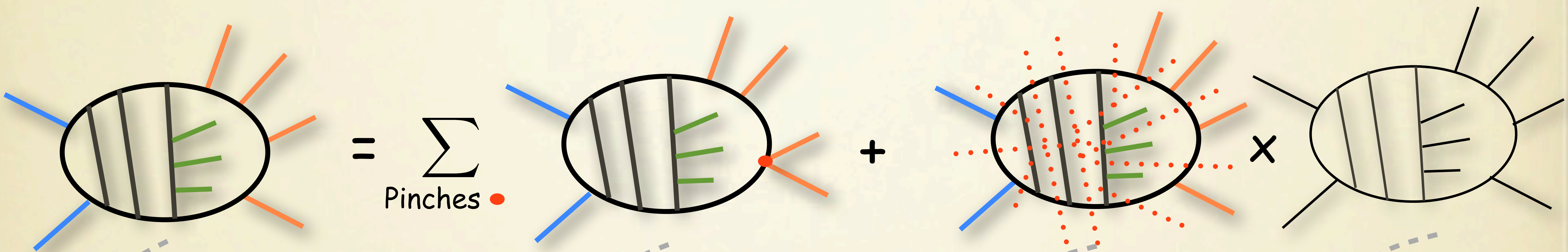
Topology		Δ	$\Delta^{\rm int}$	Δ'
$\mathcal{I}_{123457}$		184 $\{1, x_3, y_4, y_3, y_4\}$	105 $\{1, x_3, y_3, \lambda_{11}, \lambda_{22}, \lambda_{12}\}$	19 (+6) $\{1, x_3, y_3\}$
$\mathcal{I}_{134567}$		240 $\{1, x_3, x_4, y_2, y_3, y_4\}$	30 $\{1, y_2, \lambda_{11}, \lambda_{22}, \lambda_{12}\}$	4 (+2) $\{1, y_2\}$
$\mathcal{I}_{234567}$		245 $\{1, x_3, x_4, y_2, y_3, y_4\}$	137 $\{1, x_3, y_2, y_3, \lambda_{11}, \lambda_{22}, \lambda_{12}\}$	51 (+4) $\{1, x_3, y_2, y_3\}$

☑ **Arbitrary** (external and internal) **kinematics!**

The Geometry of Cut-Residues

Peraro Primo & ***P.M.***

- **ℓ-Loop Recurrence Relation**



n-line
graph

(n-1)-line
graph

coefficient
product of simpler amplitudes

Master functions

(polar coordinates)
Harmonic expansion of the residue:
Rotation Invariance manifest

Towards 2-loop Automation

- **Application of the Integration over Transverse Angles**

  - ☑ Simplifying the integrands to be reduced
    Removing the transverse direction ==> less coefficients to be determined

  - ☑ Generalising and extending to all-loop the R2-integration

Towards 2-loop Automation

- **Application of the Integration over Transverse Angles**

  - ☑ Simplifying the integrands to be reduced
    Removing the transverse direction ==> less coefficients to be determined

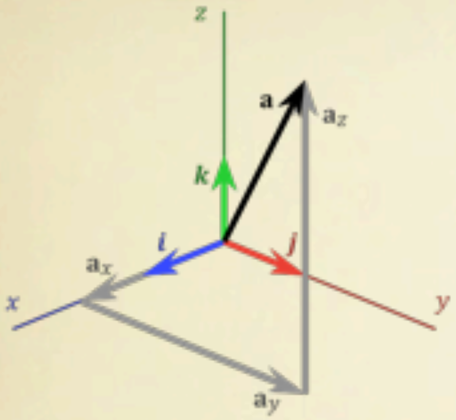
  - ☑ Generalising and extending to all-loop the R2-integration

- **Integrand Reduction + IBP-id's**

  - Improved IBP Solver   Reduze; Fire;...

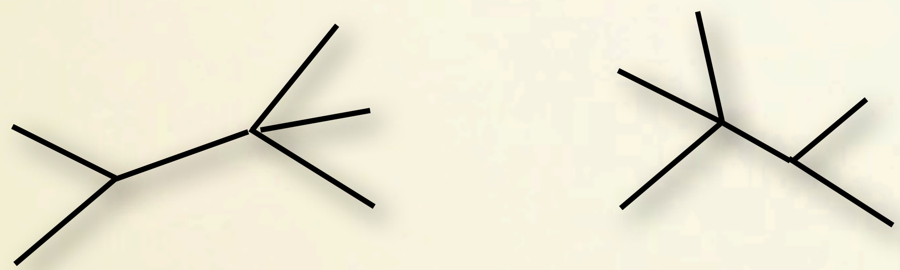
  - Algebraic Geometry Methods
    Kosower Gluza Kaida; Ita; Larsen Zhang;

>> Zhang



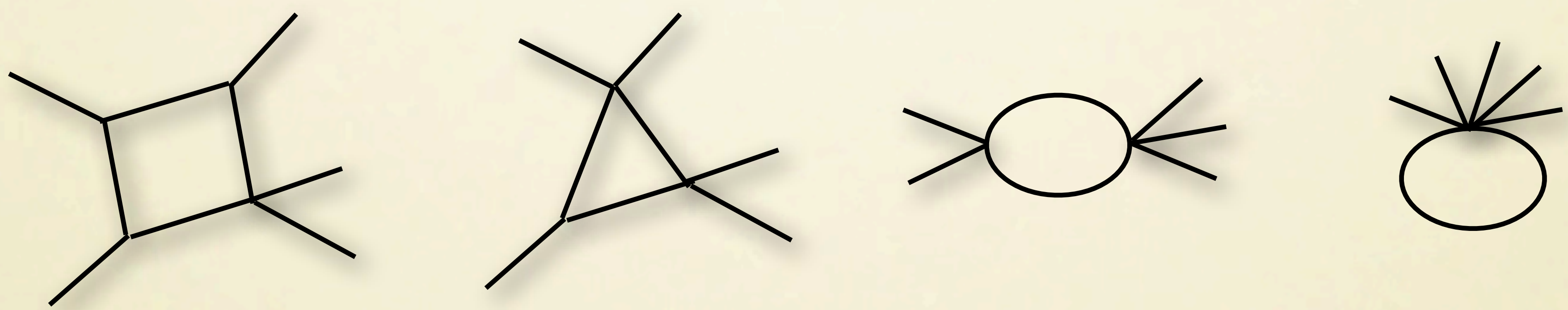
Basis :: Master Functions

- **Tree level**



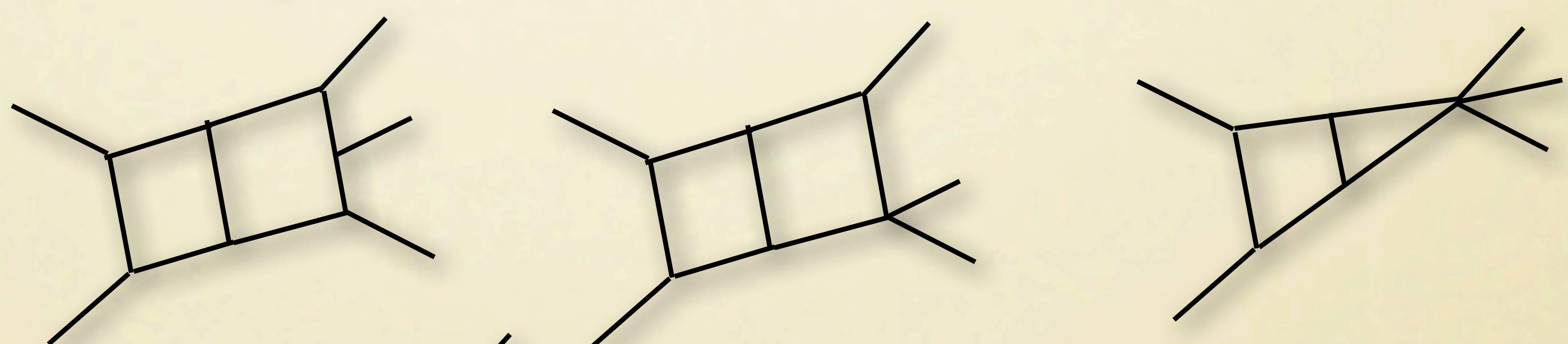
Known!

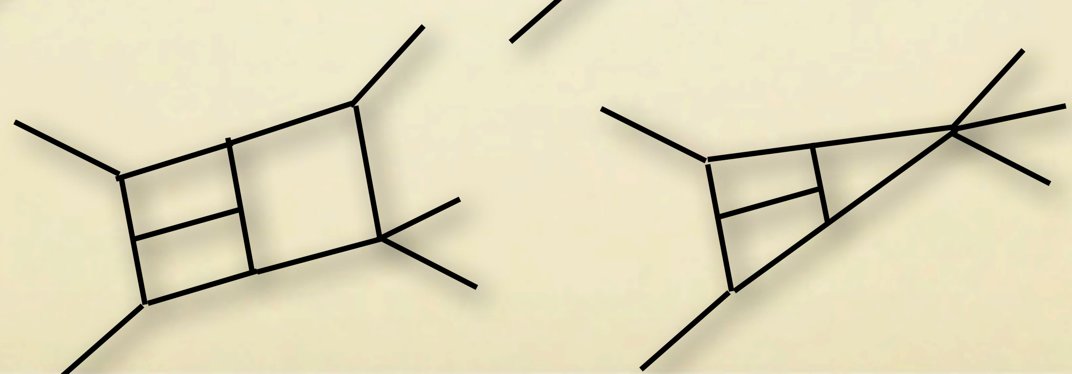
- **One Loop**



Known!

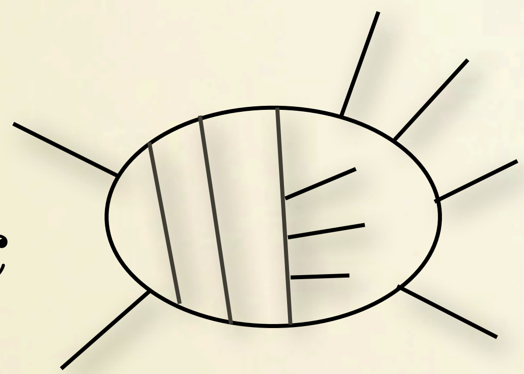
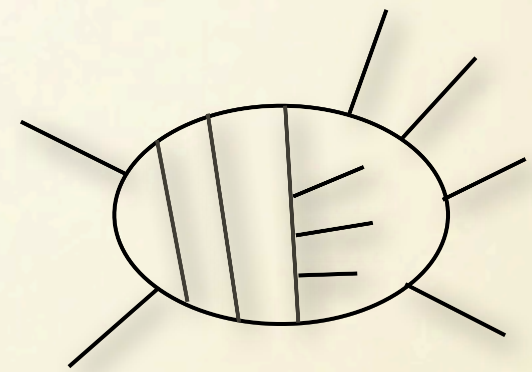
- **Higher Loops**





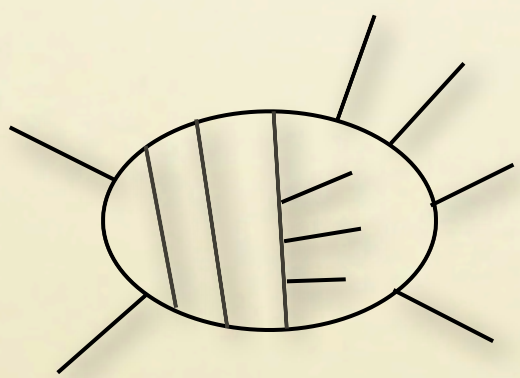
?Unknown?

Differential Equations for Master Integrals

∂_x  $= A(d, x)$ 

kinematic variable (s,t,u, masses)

space-time dimensions

 = ?

Kotikov; Remiddi;
Gehrmann Remiddi
Argeri Bonciani Ferroglia Remiddi **P.M.**
Aglietti Bonciani DeGrassi Vicini
Weinzierl
...
Henn;
Henn Smirnov & Smirnov
Henn Melnikov, Smirnov
Caron-Huot Henn
Gehrmann vonManteuffel Tancredi
Lee
Argeri diVita Mirabella Schlenk Schubert Tancredi **P.M.**
diVita Schubert Yundin **P.M.**
Papadopoulos
Papadopoulos Tommasini Wever

Quantum Mechanics

- **Schroedinger Eq'n** (**ε**-*linear Hamiltonian*)

$$i\hbar\,\partial_t|\Psi(t)\rangle = H(\epsilon,t)|\Psi(t)\rangle\ , \qquad H(\epsilon,t) = H_0(t) + \epsilon H_1(t)$$

- **Interaction Picture**

$$H_{i,I}(t) = B^\dagger(t)\ H_i(t)\ B(t)$$

- **Matrix Transform**

$$i\hbar\,\partial_t B(t) = H_0(t)B(t) \qquad B(t) = e^{-\frac{i}{\hbar}\int_{t_0}^{t} d\tau H_0(\tau)}$$

- **Schroedinger Eq'n** (*canonical form*)

$$i\hbar\,\partial_t|\Psi_I(t)\rangle = \epsilon\,H_{1,I}(t)|\Psi_I(t)\rangle\ ,$$

Magnus Expansion

Argeri, Di Vita, Mirabella,
Schlenk, Schubert, Tancredi, **P.M.** (2014)

System of 1st ODE

$$\partial_x Y(x) = A(x) Y(x) \ , \quad Y(x_0) = Y_0 \ .$$

A(x) non-commutative

solution: *Matrix Exponential*

$$Y(x) = e^{\Omega(x,x_0)}\, Y(x_0) \equiv e^{\Omega(x)}\, Y_0 \,, \qquad \Omega(x) = \sum_{n=1}^{\infty} \Omega_n(x) \ .$$

BCH-formula

$$\Omega_1(x) = \int_{x_0}^{x} d\tau_1 A(\tau_1) \ ,$$

$$\Omega_2(x) = \frac{1}{2}\int_{x_0}^{x} d\tau_1 \int_{x_0}^{\tau_1} d\tau_2 \, [A(\tau_1), A(\tau_2)] \ ,$$

$$\Omega_3(x) = \frac{1}{6}\int_{x_0}^{t} d\tau_1 \int_{x_0}^{\tau_1} d\tau_2 \int_{x_0}^{\tau_2} d\tau_3 \, [A(\tau_1),[A(\tau_2),A(\tau_3)]] + [A(\tau_3),[A(\tau_2),A(\tau_1)]] \ .$$

....

Iterated Integrals

$$\mathcal{C}^{[\gamma]}_{i_k,\ldots,i_1} \equiv \int_\gamma d\log\eta_{i_1} \ldots d\log\eta_{i_k} \equiv \int_{0\le t_1 \le \ldots \le t_k \le 1} g^\gamma_{i_k}(t_k)\ldots g^\gamma_{i_1}(t_1)\, dt_1 \ldots dt_k \qquad g^\gamma_i(t) = \frac{d}{dt}\log\eta_i(\gamma(t))$$

$$\mathcal{C}^{[\gamma]}_{\vec m}\,\mathcal{C}^{[\gamma]}_{\vec n} = \mathcal{C}^{[\gamma]}_{\vec m} \sqcup\!\sqcup\, \mathcal{C}^{[\gamma]}_{\vec n} = \sum_{\vec p = \vec m \sqcup\!\sqcup \vec n} \mathcal{C}^{[\gamma]}_{\vec p} \qquad \mathcal{C}^{[\alpha\beta]}_{i_k,\ldots,i_1} = \sum_{p=0}^{k} \mathcal{C}^{[\alpha]}_{i_k,\ldots,i_{p+1}}\, \mathcal{C}^{[\beta]}_{i_p,\ldots,i_1}$$

Chen
Goncharov
Remiddi Vermaseren
Gehrmann Remiddi
Bonciani Remiddi **P.M.**
Vollinga Weinzierl
Brown
Duhr Gangl Rhodes
....

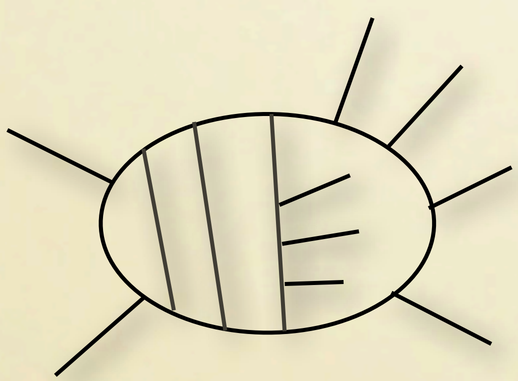
Argeri, Di Vita, Mirabella, Schlenk, Schubert, Tancredi, **P.M.** (2014)

Quantum Mechanics

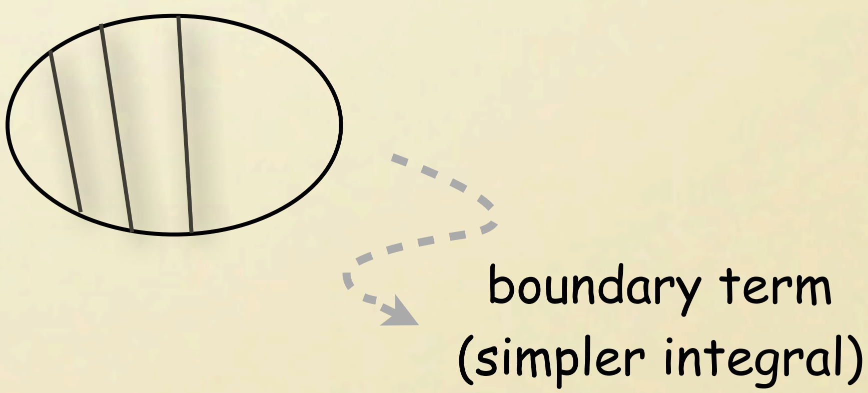
- Time-evolution in Perturbation Theory
- perturbation parameter: ε
- **Linear** Hamiltonian in ε
- **Unitary transform**
- Schroedinger Equation in the interaction picture (ε-factorization)
- solution: Dyson series

Feynman Integrals

- Kinematic-evolution in Dimensional Regularization
- space-time dimensional parameter: ε = (4-d)/2
- **Linear** system in ε
- non-Unitary **Magnus transform**
- System of Differential Equations in canonical form (ε-factorization) Henn (2013)
- solution: Dyson/Magnus series



$$= e^{\Omega(d,x)}$$



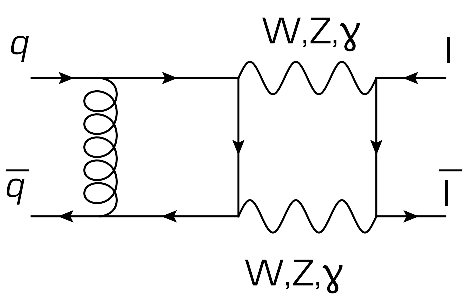
- Feynman integrals can be determined from differential equations that looks like **gauge transformations**

Drell-Yan @ 2loop EW-QCD

Bonciani, Di Vita, Schubert, **P.M.** (to appear)

$$q(p_1) + \bar{q}(p_2) \to l^-(p_3) + l^+(p_4)\,,$$
$$q(p_1) + \bar{q}'(p_2) \to l^-(p_3) + \overline{\nu}(p_4)\,.$$

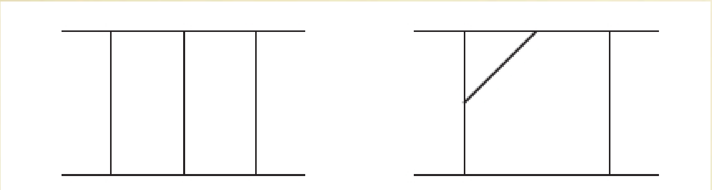
$$p_1^2 = p_2^2 = p_3^2 = p_4^2 = 0.$$



$$\frac{1}{p^2 + m_Z^2} = \frac{1}{p^2 + m_W^2 + \Delta m^2} \approx \frac{1}{p^2 + m_W^2} + \frac{\Delta m^2}{(p^2 + m_W^2)^2} + \ldots$$

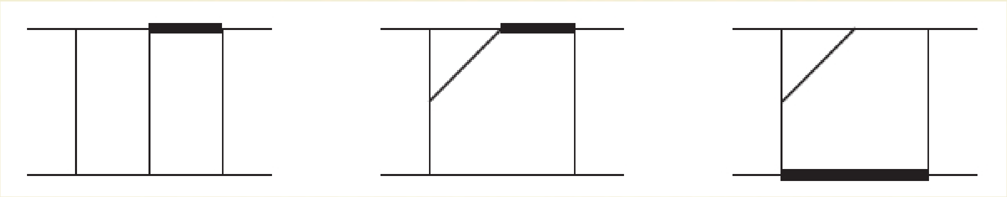
expansion is $\xi = \Delta m^2 / m_W^2$

no-mass



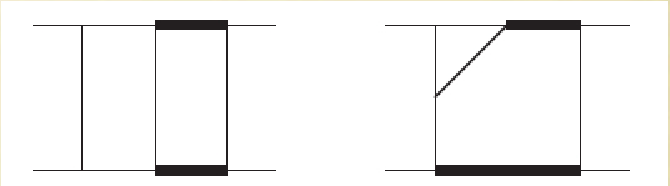
known

1-mass



new

2-mass



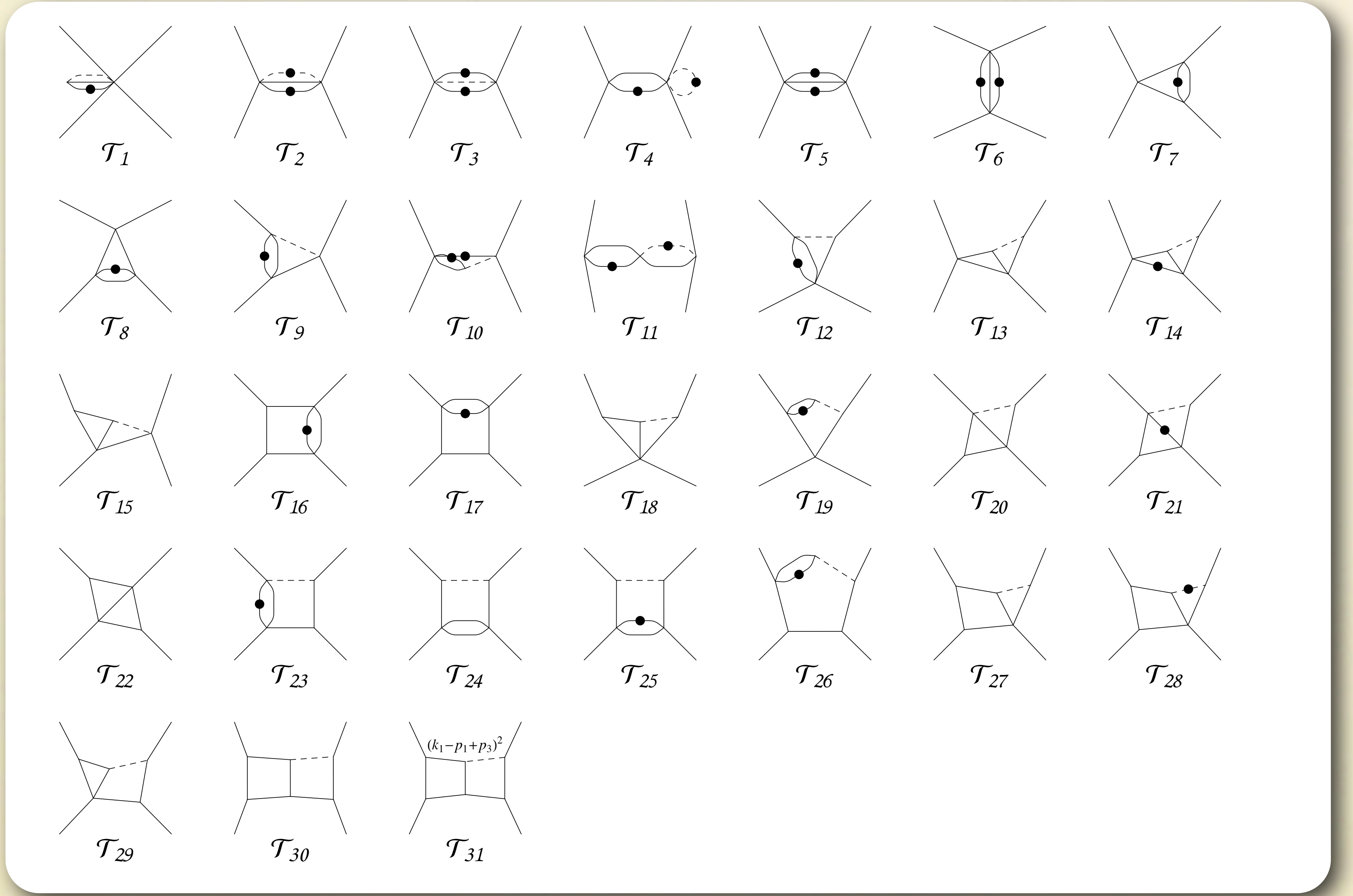
new

System of 1st ODE

$$d\mathcal{I} = \epsilon\, d\hat{A}\, \mathcal{I} \qquad \text{with} \qquad d\hat{A} = \hat{A}_x\, dx + \hat{A}_y\, dy\,.$$

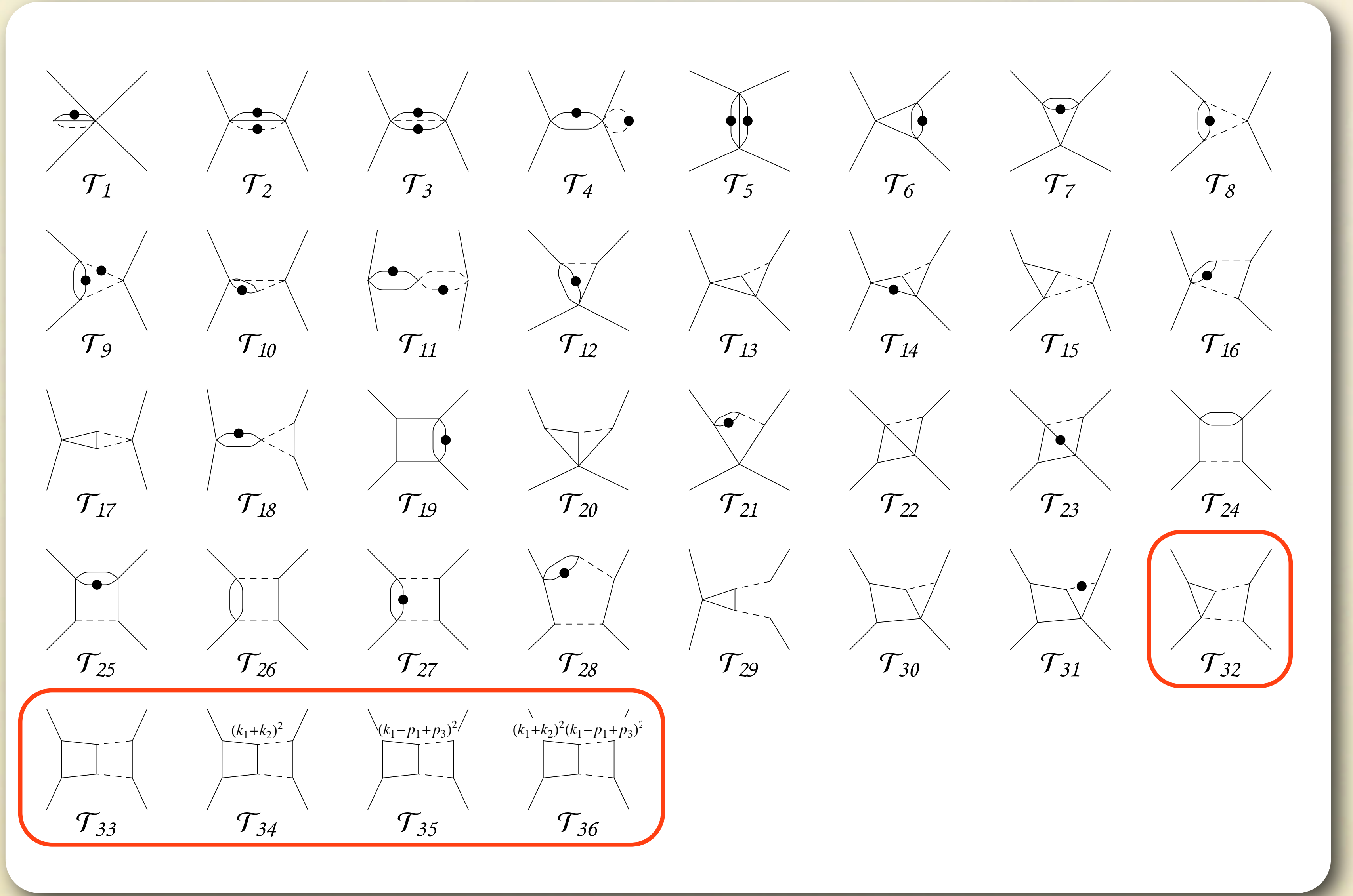
$$dA = \sum_{i=1}^{n} M_i \;\mathrm{d}\log \eta_i$$

1-Mass



- **31 MIs**
- **alphabet:** 6 rational letters
- **solution:** GPL's

2-Mass



- **36 MIs**
- **alphabet:** 12 rational + 5 irrational letters
- **solution:** Iterated integrals
  :: semi-analytic results for ○ :: numerical boundary conditions

Summary and Outlook

- IntegrANDS
  - Multi-Loop Integrand Reduction
    - Complete Development :: for generic kinematics
    - Exploiting DimReg :: Adaptive Unitarity and Transverse space integration
      - **any loop** :: we are at the same point as OPP for 1-loop.
    - Applying symmetries to the coefficients w/in the integrand decomposition
      - FDF: simple implementation of FDH scheme for generalised unitarity cuts
        Fazio, Mirabella, Torres, **PM** (2014)
      - BCJ relations @ tree-level in DimReg w/in FDF Primo, Schubert, Torres, **PM** (2015)
      - BCJ relations @ 1-Loop Chester (2016) Primo, Torres (2016)
- IntegrALS
  - Multi-Loop Master Integrals evaluation
    - Differential Equations (analytic as well as numerical) :: Magnus Exponential
      - exploiting Path invariance
      - MI's in different dimensions ==> Adaptive Differential Equations?
    - Numerical methods also very promising

Simplicity is the dawn of Discoveries

- Factorization
  - Find a region in the parameter space where the answer look simple
- Evolution algorithms :: Unitarity :: Recurrence Relation, Differential Equations, Exponentials
  - to go from simple to complex configuration

Simplicity is the dawn of Discoveries

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  - Find a region in the parameter space where the answer look simple

- Evolution algorithms :: Unitarity :: Recurrence Relation, Differential Equations, Exponentials
  - to go from simple to complex configuration

- A(nother) beautiful, simple, innocent equation

momentum k. If we set this phase to zero, it is easy to show that that the change in the polarization vector caused by a change in the reference momentum is given by:

$$\epsilon^+_\mu(p,k) \to \epsilon^+_\mu(p,k') - \sqrt{2}\frac{\langle kk'\rangle}{\langle kp\rangle\langle k'p\rangle}p_\mu. \qquad (2.23)$$

Mangano Parke

- ☑ Transversality & on-shellness
- ☑ Gauge invariance/Ward Id'y
- ☑ (holomorphic) Soft Factors
- ☑ Little Group transform
- ☑ Momentum twistors
- ☑ Color/Kinematics duality