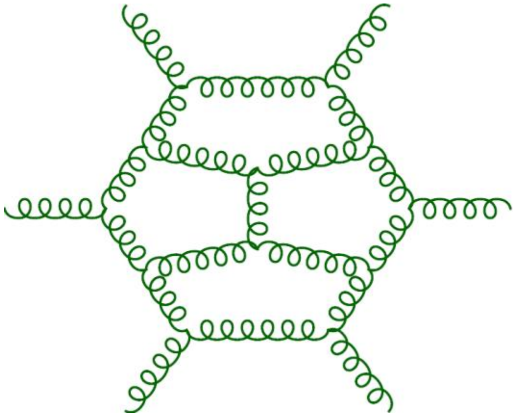
MHV Amplitudes from 0 to 5 Loops (in planar N=4 super-Yang-Mills)



Lance Dixon (SLAC)

with J. Drummond, C. Duhr, M. von Hippel, J. Pennington

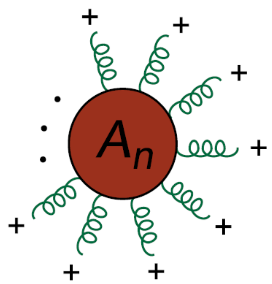
1308.2276, 1402.3300

with S. Caron-Huot, M. von Hippel, A. McLeod,

1509.08127 and to appear

MHV@30: Fermilab, March 16, 2016

The Unreasonable Simplicity of QCD Amplitudes…



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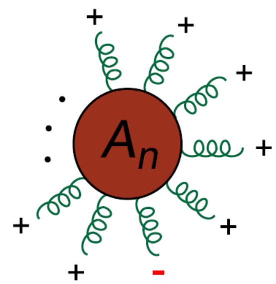

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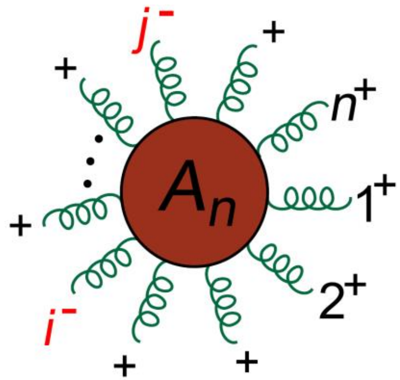
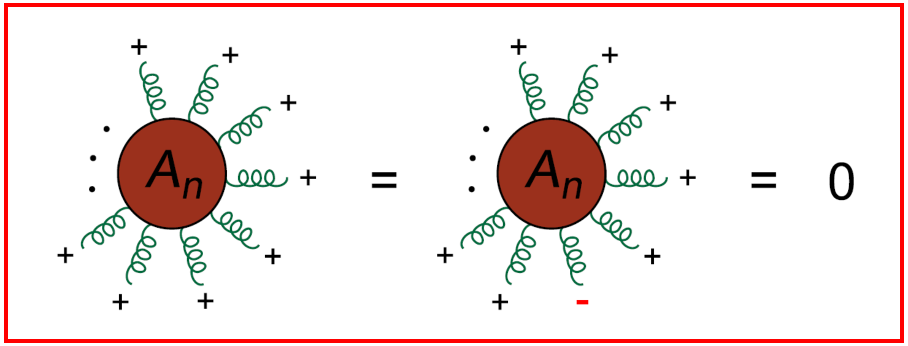


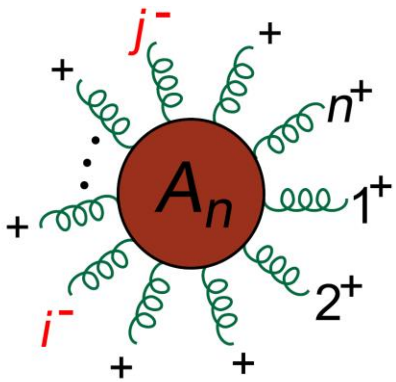
$$= \frac{\langle i\, j \rangle^4}{\langle 1\, 2 \rangle \langle 2\, 3 \rangle \cdots \langle n\, 1 \rangle}$$

Parke, Taylor (1986); Mangano, Parke, Xu (1987)

…is due to QCD's embedding into N=4 super-Yang-Mills theory



Grisaru, Pendleton, van Nieuwenhuizen (1976); Grisaru, Pendleton (1977)

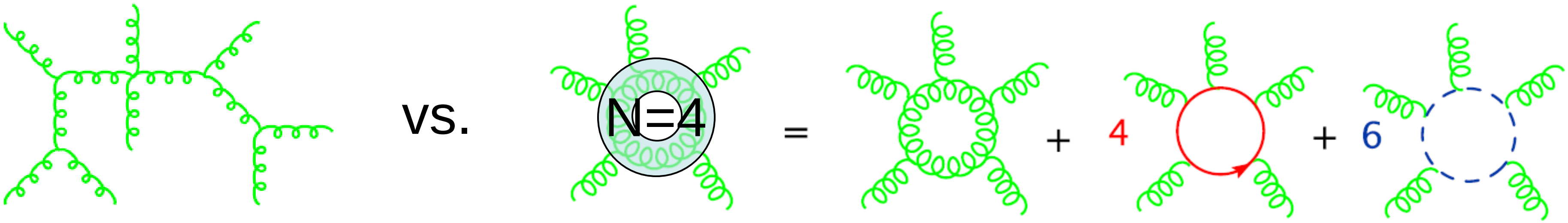


$$= \frac{\delta^8\left(\sum_{i=1}^{n} \lambda_i^\alpha \eta_i^A\right)}{\langle 1\, 2\rangle\langle 2\, 3\rangle \cdots \langle n\, 1\rangle}$$

Parke, Taylor (1985); Kunszt (1986); Nair (1988)

(N=4) SUSY Ward Identities: Hold at tree-level in QCD because for n-gluon tree amplitudes matter doesn't matter

N=4 SYM ≠ QCD at loop level

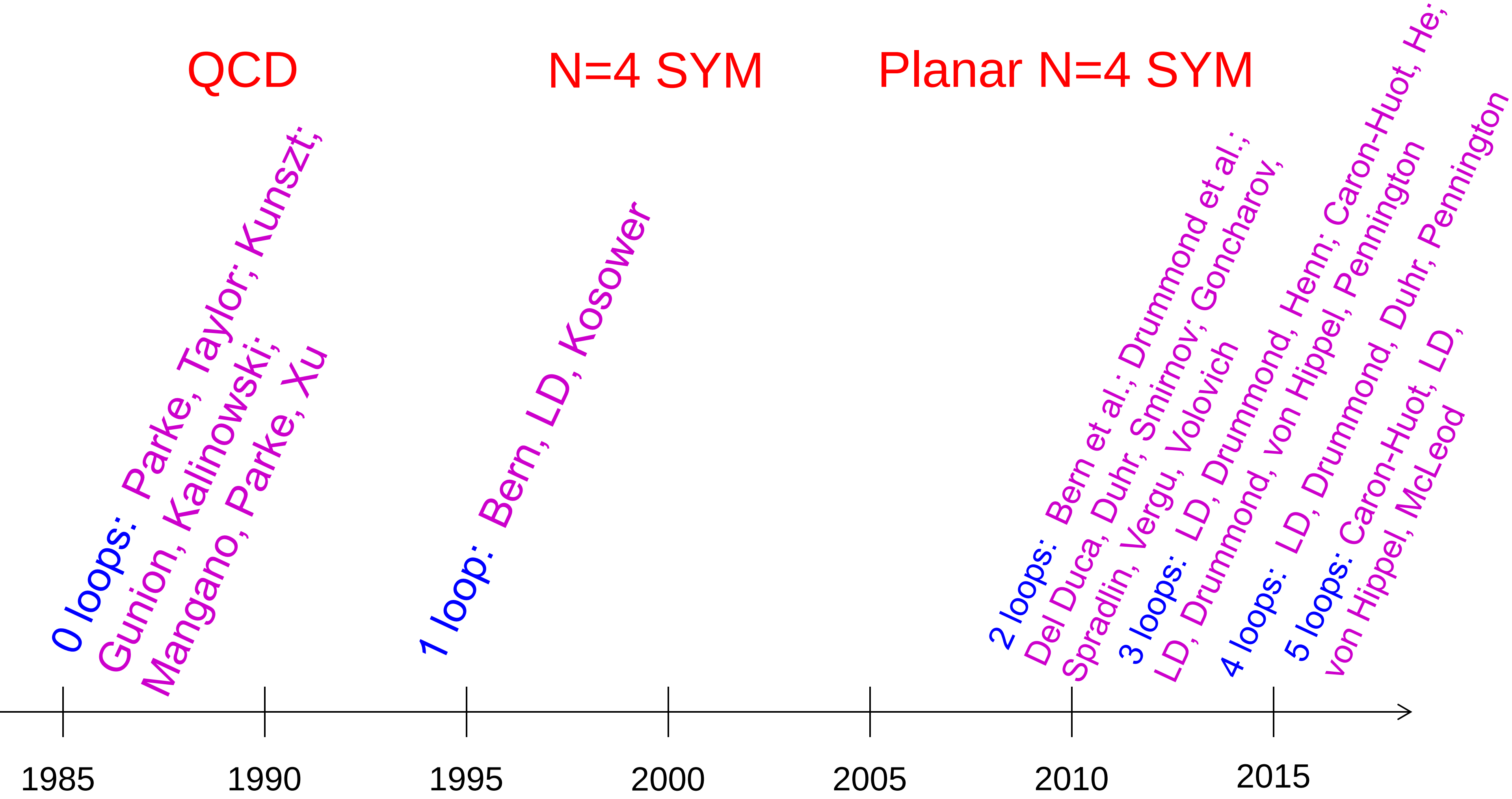


- Conformally invariant ($\beta = 0$)
- Uniform transcendental weight: " $\ln^{2L} x$ " at L loops
- Nevertheless, N=4 SYM a testing ground for methods for pQCD at colliders since 1990s:
  - All massless particles
  - IR structure perturbatively similar (dominated by gluons)
  - Factorization similar: collinear, multi-particle, Regge/BFKL limits

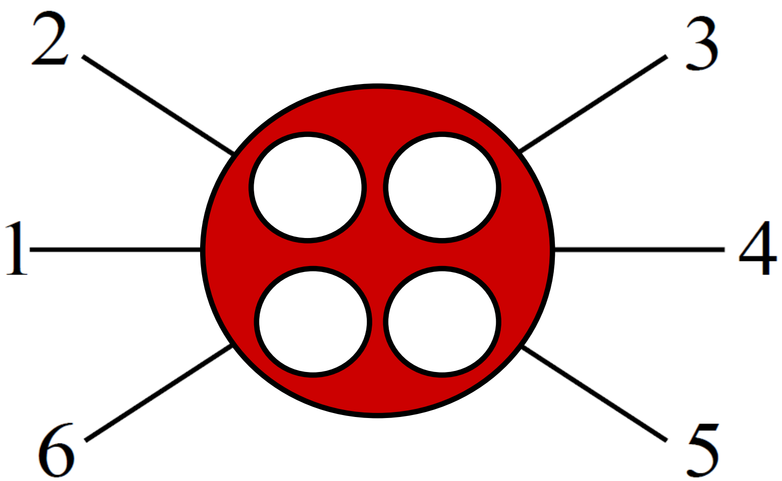
Planar (large N_c) N=4 SYM less like QCD

- Amplitudes equivalent to Wilson loops
- Dual (super)conformal invariance for any *n*
- Amplitudes for *n*=4 or 5 gluons "trivial" to all loop orders
- Strong coupling → minimal area surfaces
- Perturbation theory has finite radius of convergence (no renormalons, no instantons)
- Integrability + OPE → exact, nonperturbative predictions for near-collinear limit

MHV (6-point) timeline



Recent progress from 3 to 5 loops:
bootstrap integrated loop amplitudes directly,
without ever peeking inside the loops



Ideally, do this nonperturbatively (so no loops to peek inside) for general kinematics

The Strategy

1. Make ansatz for 6 gluon scattering amplitudes as linear combination of "hexagon functions"
2. Use dual superconformal ($\bar{Q}$ or descent) equations to prune ansatz globally in the kinematics
3. Use "boundary value data" (multi-Regge, OPE limits) to fix constants in ansatz. Constraints all linear → just solve linear equations for rational numbers in ansatz
4. Cross check.

- Works for 6-gluon amplitude, first "nontrivial" amplitude in planar N=4 SYM, through 5 loops for MHV = (--++++) [also 4 loops for NMHV = (---+++) 1509.08127]

BDS Ansatz

Bern, LD, Smirnov, hep-th/0505205

- Captures all IR divergences of amplitude
- Also accounts for an anomaly in dual conformal invariance due to IR divergence
- **Fails for** *n = 6,7,…*
- But failure (remainder function) is dual conformally invariant

$$\mathcal{A}_n^{\rm BDS} = \mathcal{A}_n^{\rm tree} \times \exp\Big[\sum_{l=1}^{\infty}\big[\frac{\lambda}{8\pi^2}\big]^l \Big(f^{(l)}(\epsilon) M_n^{(1)}(l\epsilon; s_{ij}) + C^{(l)} + \mathcal{O}(\epsilon)\Big)\Big]$$

constants, indep.of kinematics

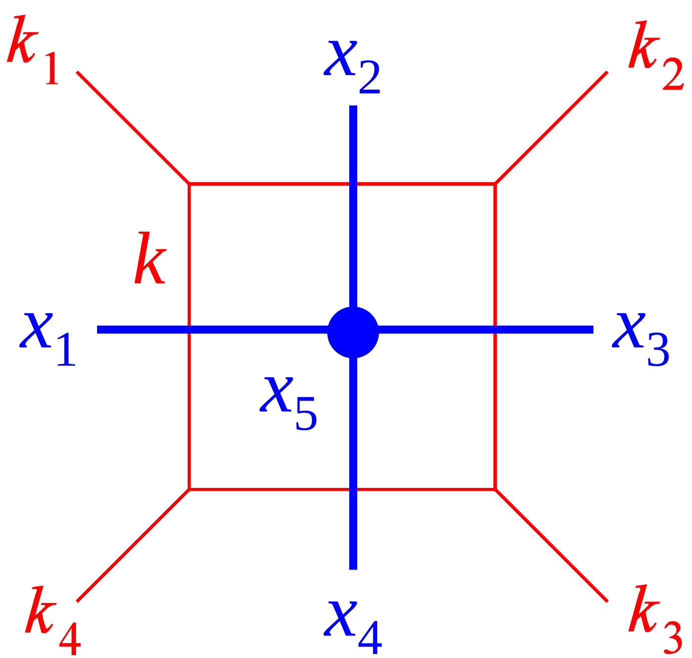
all kinematic dependence from 1-loop amplitude

Dual conformal invariance

Broadhurst (1993); Lipatov (1999); Drummond, Henn, Smirnov, Sokatchev, hep-th/0607160

Conformal symmetry acting in momentum space,
on dual or sector variables x_i

First seen in N=4 SYM planar amplitudes in the loop integrals



$$I = \int d^4k \frac{(k_1+k_2)^2(k_2+k_3)^2}{k^2(k-k_1)^2(k-k_1-k_2)^2(k+k_4)^2}$$

$$I = \int d^4x_5 \frac{x_{13}^2 x_{24}^2}{x_{15}^2 x_{25}^2 x_{35}^2 x_{45}^2}$$

$$k_1 = x_{12}$$
$$k_2 = x_{23}$$
$$k_3 = x_{34}$$
$$k_4 = x_{41}$$
$$k = x_{15}$$

invariant under inversion:

$$x_i^\mu \to \frac{x_i^\mu}{x_i^2}$$

$$x_{ij}^2 \to \frac{x_{ij}^2}{x_i^2 x_j^2}, \qquad \mathrm{d}^4 x_i \to \frac{\mathrm{d}^4 x_i}{x_i^8}$$

Dual conformal invariance (cont.)

- Amplitude fixed, up to functions of dual conformally invariant cross ratios:

$$u_{ijkl} \equiv \frac{x_{ij}^2 x_{kl}^2}{x_{ik}^2 x_{jl}^2}$$

$$x_{ij}^2 = (k_i + k_{i+1} + \cdots + k_{j-1})^2$$

- $x_{i,i+1}^2 = k_i^2 = 0$ → no such variables for $n = 4,5$

$n = 6$ → precisely 3 ratios:

$$u_1 = \frac{x_{13}^2 x_{46}^2}{x_{14}^2 x_{36}^2} = \frac{s_{12}s_{45}}{s_{123}s_{345}}$$

+ 2 cyclic perm's

From 9 variables to just 3:

$s_{12}, s_{23}, s_{34}, s_{45}, s_{56}, s_{61}, s_{123}, s_{234}, s_{345}$

$\rightarrow u_1, u_2, u_3$

Remainder function, starts at 2 loops

$$\mathcal{A}_6^{\text{MHV}}(\epsilon; s_{ij}) = \mathcal{A}_6^{\text{BDS}}(\epsilon; s_{ij}) \exp[R_6(u_1, u_2, u_3)]$$

A better quantity

$$\mathcal{E}(u,v,w) = \exp\left[R_6 - \frac{\gamma_K(a)}{8} Y\right] \equiv \frac{\mathcal{A}_6^{\rm MHV}}{\mathcal{A}_6^{\rm BDS-like}}$$

where $a = \frac{\lambda}{8\pi^2}$ $\gamma_K(a) = 4 f_0(a)$ cusp anomalous dimension

$$Y(u,v,w) \equiv \mathrm{Li}_2(1-u) + \mathrm{Li}_2(1-v) + \mathrm{Li}_2(1-w) + \frac{1}{2}\left(\ln^2 u + \ln^2 v + \ln^2 w\right)$$

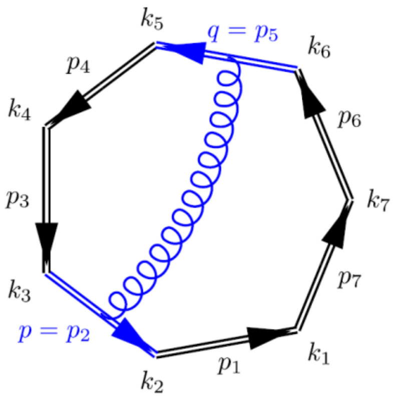
$$\frac{\mathcal{A}_6^{\rm BDS-like}}{\mathcal{A}_6^{\rm MHV\,(0)}} = \exp\left[\sum_{L=1}^{\infty} a^L \left(f^{(L)}(\epsilon)\frac{1}{2}\hat{M}_6(L\epsilon) + C^{(L)}\right)\right]$$

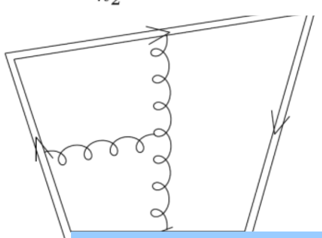
$$\begin{aligned}\hat{M}_6(\epsilon) &= M_6^{1-\rm loop} + Y(u,v,w) \\ &= \sum_{i=1}^{6}\left[-\frac{1}{\epsilon^2}\Big(1-\epsilon\ln(-s_{i,i+1})\Big) - \ln(-s_{i,i+1})\ln(-s_{i+1,i+2}) + \frac{1}{2}\ln(-s_{i,i+1})\ln(-s_{i+3,i+4})\right] \\ &\quad + 6\,\zeta_2\,,\end{aligned}$$

No 3-particle invariants

MHV Amplitudes = Wilson Loops

Motivated by strong-coupling correspondence Alday, Maldacena, 0705.0303





- One loop, *n=4* Drummond, Korchemsky, Sokatchev, 0707.0243
- One loop, any *n* Brandhuber, Heslop, Travaglini, 0707.1153
- Two loops, *n=4,5,6* Drummond, Henn, Korchemsky, Sokatchev, 0709.2368, 0712.1223, 0803.1466; Bern, LD, Kosower, Roiban, Spradlin, Vergu, Volovich, 0803.1465

- Wilson-loop VEV **always matches** [MHV] scattering amplitude!
- Justifies dual conformal invariance for amplitude DHKS, 0712.1223
- Twistor action → duality: e.g. Adamo, Bullimore, Mason, Skinner, 1104.2890

Two loop answer: $R_6^{(2)}(u_1,u_2,u_3)$

- Simplified to classical polylogarithms by
Goncharov, Spradlin, Vergu, Volovich, 1006.5703

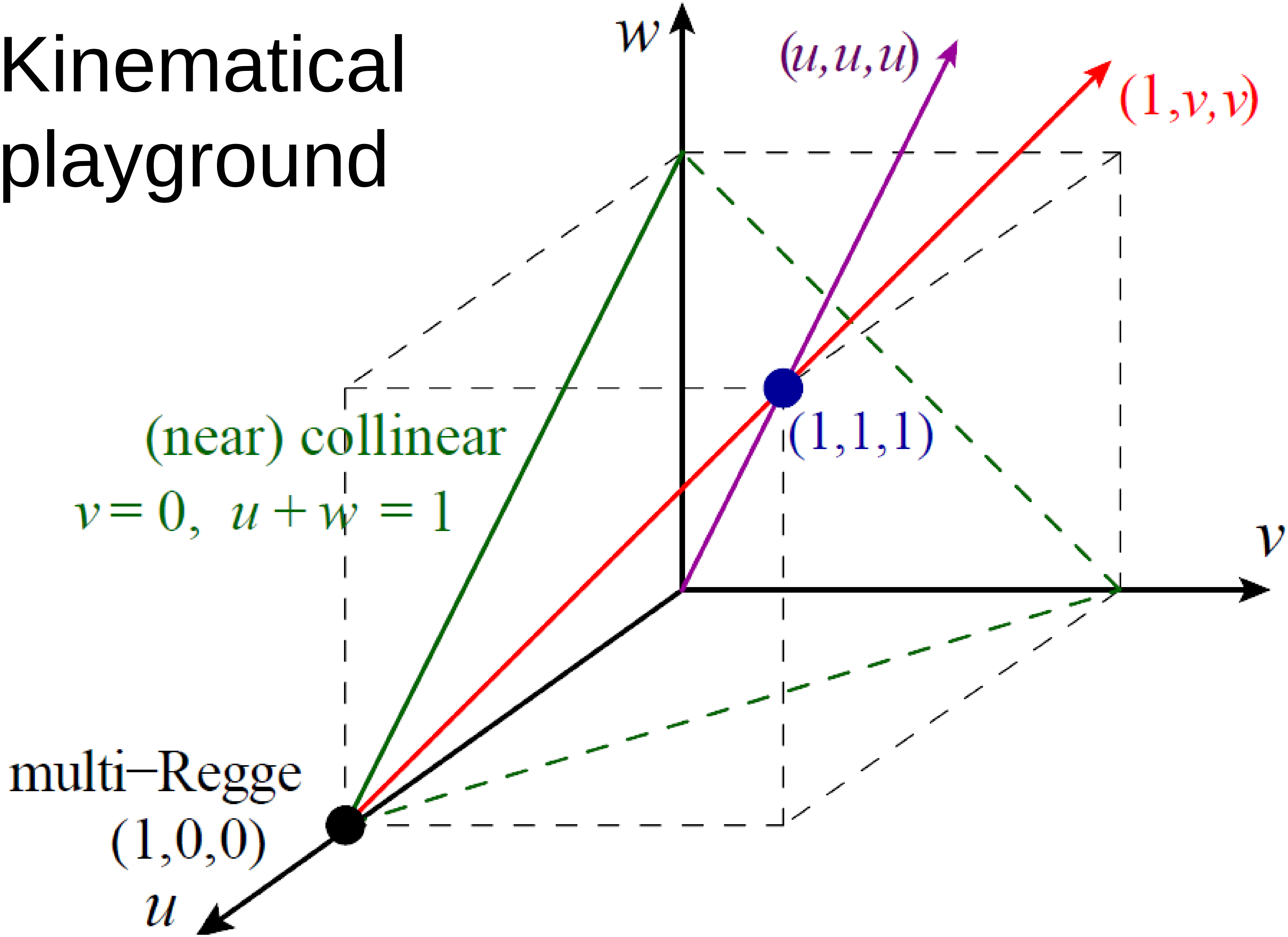
$$R_6^{(2)}(u_1,u_2,u_3) = \sum_{i=1}^{3}\left(L_4(x_i^+,x_i^-) - \frac{1}{2}\,\mathrm{Li}_4(1-1/u_i)\right) - \frac{1}{8}\left(\sum_{i=1}^{3}\mathrm{Li}_2(1-1/u_i)\right)^2 + \frac{1}{24}J^4 + \frac{\pi^2}{12}J^2 + \frac{\pi^4}{72}$$

$$L_4(x^+,x^-) = \frac{1}{8!!}\log(x^+x^-)^4 + \sum_{m=0}^{3}\frac{(-1)^m}{(2m)!!}\log(x^+x^-)^m(\ell_{4-m}(x^+) + \ell_{4-m}(x^-))$$

$$\ell_n(x) = \frac{1}{2}\left(\mathrm{Li}_n(x) - (-1)^n\,\mathrm{Li}_n(1/x)\right) \qquad J = \sum_{i=1}^{3}(\ell_1(x_i^+) - \ell_1(x_i^-))$$

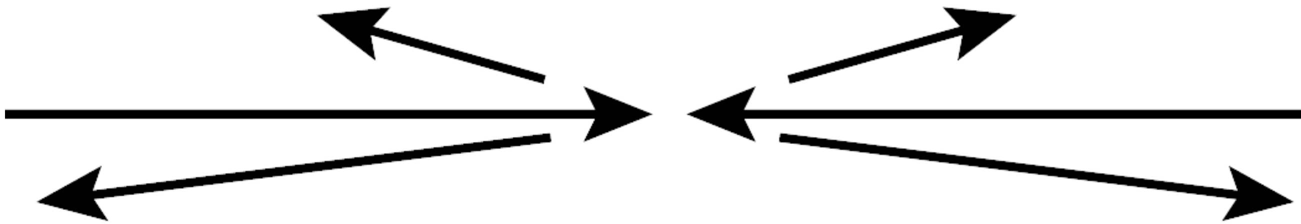
$$x_i^\pm = u_i x^\pm, \qquad x^\pm = \frac{u_1+u_2+u_3-1\pm\sqrt{\Delta}}{2u_1u_2u_3} \qquad \Delta = (u_1+u_2+u_3-1)^2 - 4u_1u_2u_3$$

Kinematical playground



Multi-Regge limit

• Minkowski kinematics, large rapidity separations between the 4 final-state gluons:

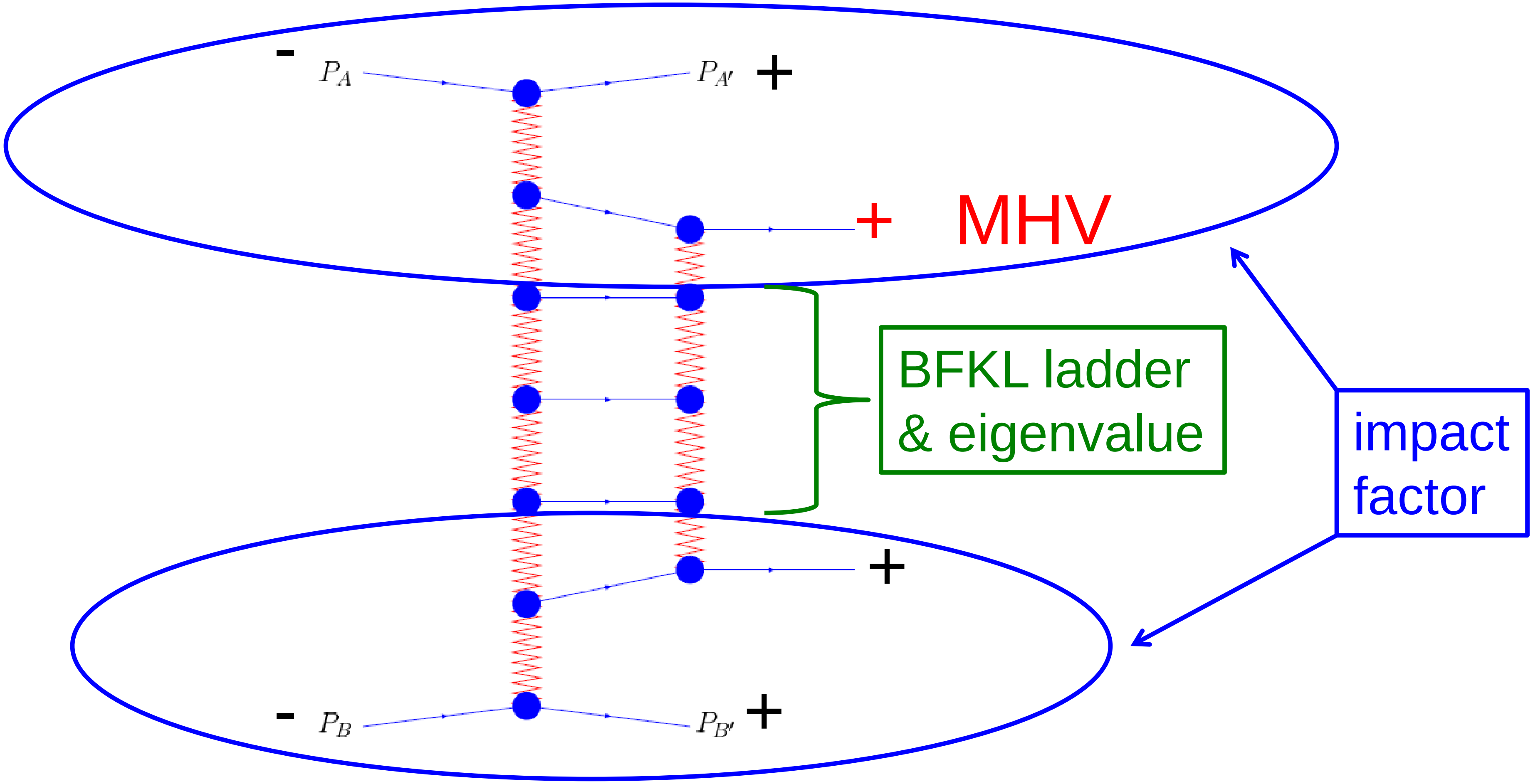


• Properties of planar N=4 SYM amplitude in this limit studied extensively at weak coupling:

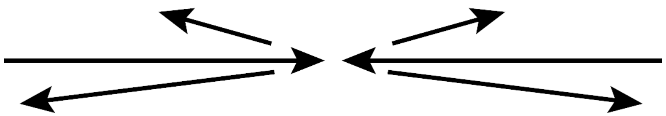
Bartels, Lipatov, Sabio Vera, 0802.2065, 0807.0894; Lipatov, 1008.1015; Lipatov, Prygarin, 1008.1016, 1011.2673; Bartels, Lipatov, Prygarin, 1012.3178, 1104.4709; LD, Drummond, Henn, 1108.4461; Fadin, Lipatov, 1111.0782; LD, Duhr, Pennington, 1207.0186; Basso, Caron-Huot, Sever, 1407.3766

2 → 4 Multi-Regge picture

Bartels, Lipatov, Sabio Vera, 0802.2065



2→4 multi-Regge limit



- Euclidean MRK limit vanishes
- To get nonzero result for physical region, first let

$u_1 \to u_1 e^{-2\pi i}$, then $u_1 \to 1$, $u_2, u_3 \to 0$

$$\frac{u_2}{1-u_1} \to \frac{1}{|1-z|^2} \qquad \frac{u_3}{1-u_1} \to \frac{|z|^2}{|1-z|^2}$$

$$R_6^{(L)} \to (2\pi i) \sum_{r=0}^{L-1} \ln^r(1-u) \left[g_r^{(L)}(z,\bar{z}) + 2\pi i\, h_r^{(L)}(z,\bar{z}) \right]$$

$g_r^{(L)}$ and $h_r^{(L)}$ all well understood by now

Fadin, Lipatov, 1111.0782; LD, Duhr, Pennington, 1207.0186; Pennington, 1209.5357; Basso, Caron-Huot, Sever, 1407.3766;

OPE Limits

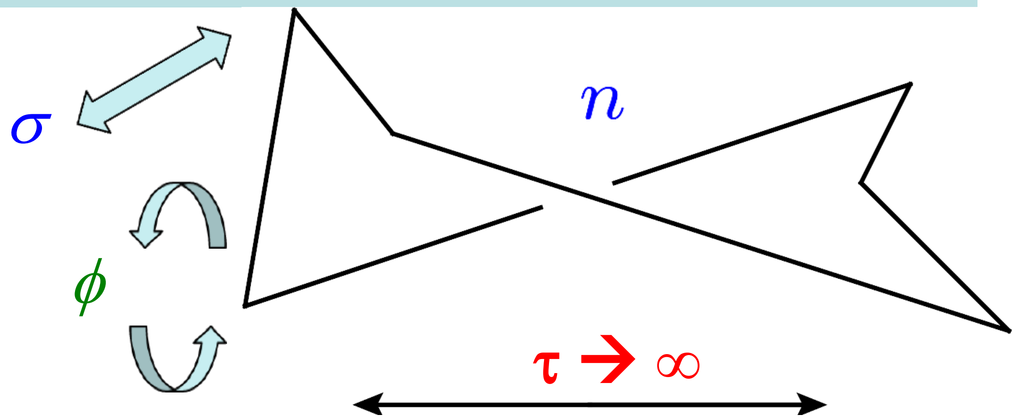
Alday, Gaiotto, Maldacena, Sever, Vieira, 1006.2788; GMSV, 1010.5009; 1102.0062
Basso, Sever, Vieira [BSV], 1303.1396; 1306.2058; 1402.3307, 1407.1736;
Basso, Caetano, Córdova, Sever, Vieira, 1412.1132;
Belitsky, 1407.2853, 1410.2534, 1506.02598;
Drummond, Papathanasiou, 1507.08982;…

- $R_6^{(L)}(u,v,w)$ vanishes as $v = 1/\cosh^2\tau \to 0$ $\tau \to \infty$

Near-collinear limit (power-suppressed terms in v) described by OPE with generic form

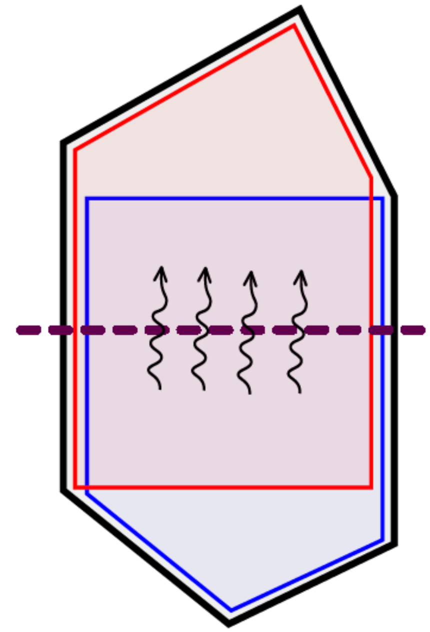
$$R_6^{(L)}(u,v,w) = R_6^{(L)}(\tau,\sigma,\phi) \sim \int dn\ C_n(g)\ \exp[-E_n(g)\tau]$$

$$u = \frac{e^\sigma \sinh\tau \tanh\tau}{2(\cosh\sigma\ \cosh\tau + \cos\phi)}$$
$$v = \frac{1}{\cosh^2\tau}$$
$$w = u\, e^{-2\sigma}$$



OPE Limits (cont.)

- OPE dominated by low-lying excitations of a flux tube
- BSV use power of integrability to determine pentagon transitions **EXACTLY** in the coupling.
- From this one can easily compute at **ANY** loop order the leading-twist one flux-tube excitation terms ($T = e^{-\tau}$):



$$T\, e^{\pm i\phi}\ [\ln T]^k f_k(\sigma), \quad k = 0,1,2,\ldots,L\text{-}1$$

the sub-leading twist, two flux-tube excitation terms

$$T^2 \{e^{\pm 2i\phi},1\}\ [\ln T]^k f_k(\sigma), \quad k = 0,1,2,\ldots,L\text{-}1$$ etc.

Basic MHV bootstrap assumption

$R_6^{(L)}(u,v,w)$, or better, $\mathcal{E}^{(L)}(u,v,w)$
is a linear combination of weight $2L$
hexagon functions at any loop order L

Functional interlude

Chen; Goncharov; Brown; …

- Multiple polylogarithms, or n-fold iterated integrals, or weight n pure transcendental functions f.
- Define by derivatives:

$$d\,f = \sum_{s_k \in \mathcal{S}} f^{s_k}\, d \ln s_k$$

S = finite set of rational expressions, "symbol letters", and

$$f^{s_k} \equiv \{n-1, 1\} \text{ coproduct component}$$

Duhr, Gangl, Rhodes, 1110.0458

are also pure functions, weight n-1

- Iterate: $d\,f^{s_k} \Rightarrow f^{s_j, s_k} \equiv \{n-2, 1, 1\}$ component

- Symbol = {1,1,…,1} component (maximally iterated)

Harmonic Polylogarithms of one variable (HPLs {0,1})

Remiddi, Vermaseren, hep-ph/9905237

- Subsector of hexagon functions.
- Generalize classical polylogs, $\mathrm{Li}_n(u) = \int_0^u \frac{dt}{t} \mathrm{Li}_{n-1}(t)$
- Define by iterated integration:

$$H_{0,\vec{w}}(u) = \int_0^u \frac{dt}{t} H_{\vec{w}}(t), \quad H_{1,\vec{w}}(u) = \int_0^u \frac{dt}{1-t} H_{\vec{w}}(t)$$

- Or by derivatives

$$dH_{0,\vec{w}}(u) = H_{\vec{w}}(u)\ d\ln u \quad dH_{1,\vec{w}}(u) = -H_{\vec{w}}(u) d\ln(1-u)$$

- Symbol letters: $\mathcal{S} = \{u, 1-u\}$

Hexagon function symbol letters

- Momentum twistors $Z_i^A,\ i=1,2,\ldots,6$ transform simply under dual conformal transformations. Hodges, 0905.1473
- Construct 4-brackets $\varepsilon_{ABCD} Z_i^A Z_j^B Z_k^C Z_l^D \equiv \langle ijkl \rangle$
- 15 projectively invariant combinations of 4-brackets can be factored into 9 basic ones:

$$\mathcal{S} = \{u, v, w, 1-u, 1-v, 1-w, y_u, y_v, y_w\}$$

$$u = \frac{\langle 6123\rangle\langle 3456\rangle}{\langle 6134\rangle\langle 2356\rangle} \qquad 1-u = \frac{\langle 6135\rangle\langle 2346\rangle}{\langle 6134\rangle\langle 2356\rangle}$$

$$y_u = \frac{\langle 1345\rangle\langle 2456\rangle\langle 1236\rangle}{\langle 1235\rangle\langle 3456\rangle\langle 1246\rangle}$$

+ cyclic

Hexagon function symbol letters (cont.)

- y_i not independent of u_i :

$$y_u \equiv \frac{u - z_+}{u - z_-} \,, \ \ldots \quad \text{where}$$

$$z_\pm = \frac{1}{2}\Big[-1 + u + v + w \pm \sqrt{\Delta}\Big]$$

$$\Delta = (1 - u - v - w)^2 - 4uvw$$

- Function space graded by parity:

$$\begin{array}{rcl} i\sqrt{\Delta} & \leftrightarrow & -i\sqrt{\Delta} \\ z_+ & \leftrightarrow & z_- \\ y_i & \leftrightarrow & 1/y_i \\ u_i & \leftrightarrow & u_i \end{array}$$

Branch cut condition

- All massless particles → all branch cuts start at origin in

$$s_{i,i+1}, \quad s_{i,i+1,i+2}$$

→ Branch cuts all start from 0 or ∞ in

$$u = \frac{s_{12}^2 s_{45}^2}{s_{123}^2 s_{345}^2} \qquad \text{or } v \text{ or } w$$

→ First symbol entry $\in \{u, v, w\}$ GMSV, 1102.0062

- **Powerful constraint:** At weight 8 (four loops) we would have roughly 1,675,553 functions without it; exactly 6,916 with it.

Constructing hexagon functions iteratively

{n-1,1} coproduct characterizes first derivatives

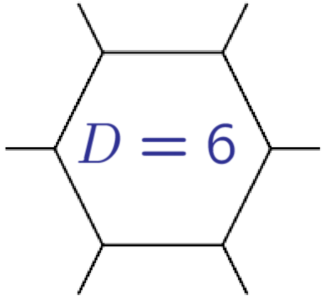
$$\left.\frac{\partial F}{\partial u}\right|_{v,w} = \frac{F^u}{u} - \frac{F^{1-u}}{1-u} + \frac{1-u-v-w}{u\sqrt{\Delta}} F^{y_u} + \frac{1-u-v+w}{(1-u)\sqrt{\Delta}} F^{y_v} + \frac{1-u+v-w}{(1-u)\sqrt{\Delta}} F^{y_w}$$

$$\frac{\partial \ln y_u}{\partial u} \nearrow$$

- Defines F up to overall constant (subject to integrability)
- Always stay in space of functions with good branch cuts
- Integrate numerically or via multiple polylogarithms $G(\ldots)$
- Or solve analytically in special limits, e.g.:

1. Multi-regge limit
2. Near-collinear (OPE) limit

The first true hexagon function



$$\Rightarrow \quad \tilde{\Phi}_6(u,v,w)$$

- Weight 3, totally symmetric in $\{u,v,w\}$ (secretly Li_3's)
- First parity odd function, so:

$$\tilde{\Phi}_6^u = \tilde{\Phi}_6^v = \tilde{\Phi}_6^w = \tilde{\Phi}_6^{1-u} = \tilde{\Phi}_6^{1-v} = \tilde{\Phi}_6^{1-w} = 0$$

- Only independent {2,1} coproduct:

$$\tilde{\Phi}_6^{y_u} = -\Omega^{(1)}(v,w,u) = -H_2^u - H_2^v - H_2^w - \ln v \, \ln w + 2\,\zeta_2$$

$$H_2^u = \mathrm{Li}_2(1-u)$$

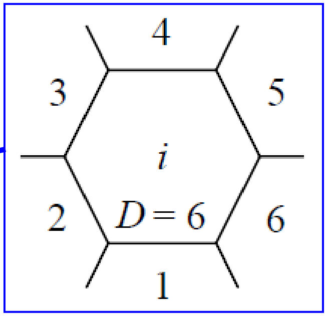
- Encapsulates first order differential equation found earlier

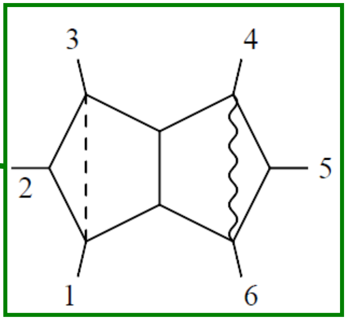
LD, Drummond, Henn, 1104.2787

How many hexagon functions?

Irreducible (non-product) ones:

Weight	y^0	y^1	y^2	y^3	y^4	y^5	y^6
1	3	-	-	-	-	-	-
2	3	-	-	-	-	-	-
3	6	1	-	-	-	-	-
4	9	3	3	-	-	-	-
5	18	4	13	6	-	-	-
6	27	4	27	29	18	-	-
7	54	4	41	63	108	39	-
8	90	4	50	108	306	238	114





many related by S_3 symmetry

What's left?

- Enumerate all hexagon functions (obeying a $\bar{Q}$ equation) with weight ***2L*** and correct symmetries
- Write most general linear combination with unkown rational-number coefficients
- Impose a series of physical constraints until all coefficients uniquely determined

Simple constraints on $\boldsymbol{R_6}$

- S_3 permutation **symmetry** in $\{u, v, w\}$
- Even under "**parity**"
- Vanishing in **collinear** limit

$$v \to 0 \qquad u + w \to 1$$

Dual superconformal invariance

- Super Wilson-loops are (dual) superconformally invariant.
- But generator $\bar{Q}$, a first-order differential operator, has an anomaly due to virtual collinear singularities:

$$\bar{Q}\, A^{(L)}{}_{n,k} \sim \partial\, A^{(L)}{}_{n,k} \sim \int A^{(L\text{-}1)}{}_{n+1,k+1}$$

Caron-Huot, 1105.5606; Bullimore, Skinner, 1112.1056; Caron-Huot, He, 1112.1060

- In some directions the source term vanishes → all loop order differential constraints

$\bar{Q}$ equation for MHV

- Constraint on first derivative of remainder function R_6 or $\mathcal{E}(u,v,w)$ has simple form
- In terms of the final entry of symbol, restricts to 6 of 9 possible letters:

$$\left\{\frac{u}{1-u}, \frac{v}{1-v}, \frac{w}{1-w}, y_u, y_v, y_w\right\}$$

- In terms of {n-1,1} coproducts, equivalent to:

$$\mathcal{E}^u + \mathcal{E}^{1-u} = \mathcal{E}^v + \mathcal{E}^{1-v} = \mathcal{E}^w + \mathcal{E}^{1-w} = 0$$

$\bar{Q}$ next-to-final-entry relations

- Further homogeneous constraints on second derivatives, or {n-2,1,1} coproducts
  – only simple in terms of $\mathcal{E}$:

$$\mathcal{E}^{y_u,u} - \mathcal{E}^{y_v,w} - \mathcal{E}^{u,y_v} - \mathcal{E}^{1-w,y_w} = 0$$

$$\mathcal{E}^{y_u,y_u} - \mathcal{E}^{y_u,y_v} - \mathcal{E}^{y_u,y_w} + \mathcal{E}^{y_v,y_w} + \mathcal{E}^{v,w} = 0$$

plus all permutations

These equations prune the space of hexagon functions – even before it is constructed!

Fixing parameters in $R_6^{(L)}$ or $\mathcal{E}^{(L)}$

Constraint	$L=1$	$L=2$	$L=3$	$L=4$	$L=5$
0. Even integrable functions	10	82	639	5153	?????
1. S_3 symmetry in u, v, w	4	23	152	1085	????
2. $\bar{Q}$ 6-final-entry condition	2	8	49	344	????
3. $\bar{Q}$ next-to-final-entry	2	7	33	156	815
4. Collinear vanishing	0	0	1	3	41
5. LL multi-Regge kinematics	0	0	0	1	25
6. NLL MRK	0	0	0	0	3
7. NNLL MRK	0	0	0	0	0

- No OPE information needed at all now!
- Have already checked $T^2 \times e^{\pm 2i\phi}$ terms through 5 loops

Numerical results

- Plot perturbative coefficients on various lines and surfaces
- Instructive to take ratios of successive loop orders $R_6^{(L)}/R_6^{(L-1)} = \bar{R}_6^{(L)}$
  - Planar N=4 SYM has no instantons and no renormalons.
  - Perturbative expansion has finite radius of convergence, 1/8
  - For "asymptotically large orders", $R_6^{(L)}/R_6^{(L-1)}$ should approach -8

Cusp anomalous dimension $\gamma_K(\lambda)$

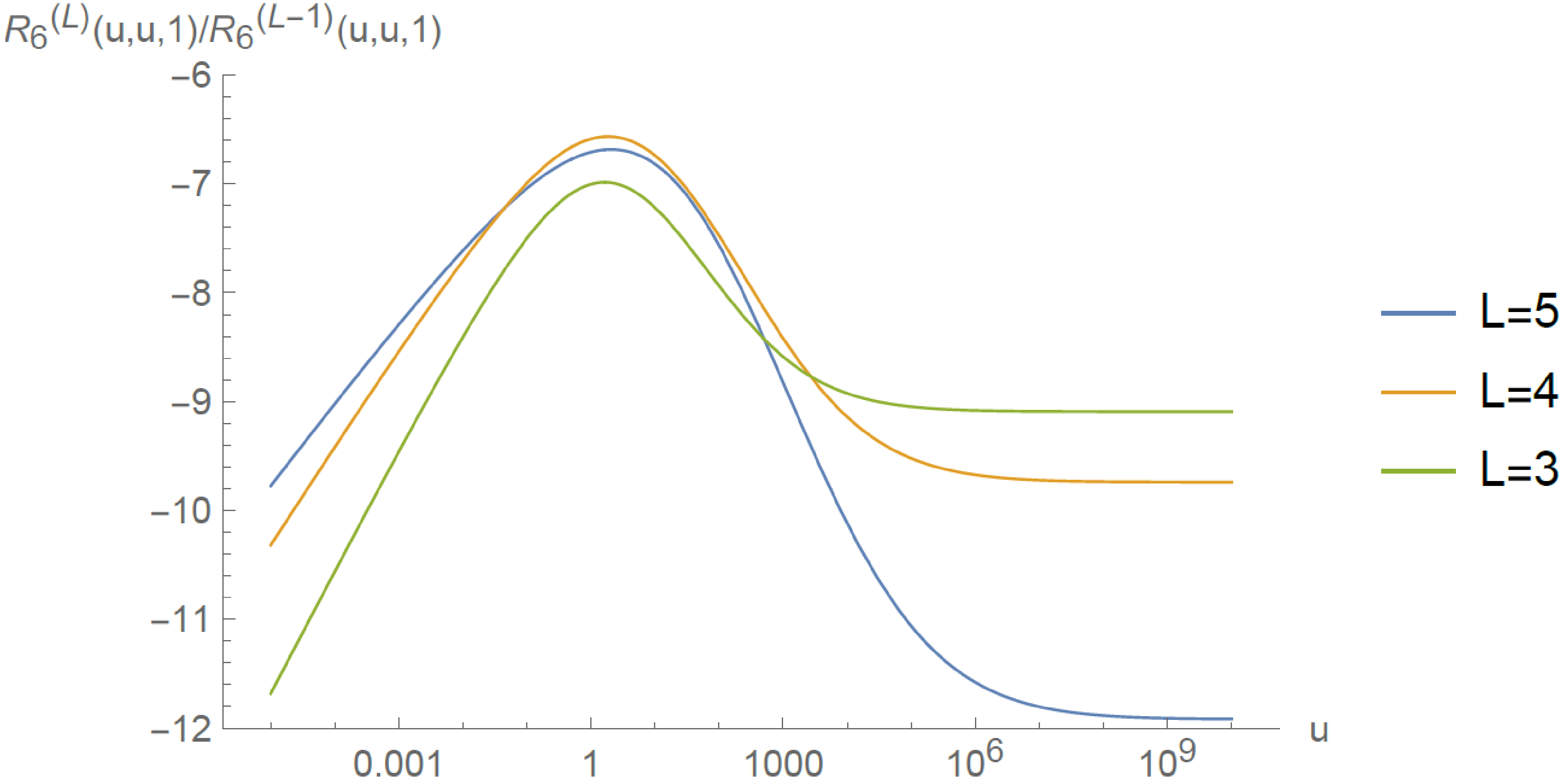
- Known to all orders, Beisert, Eden, Staudacher [hep-th/0610251]

closely related to amplitude/Wilson loop, use as benchmark for approach to large orders:

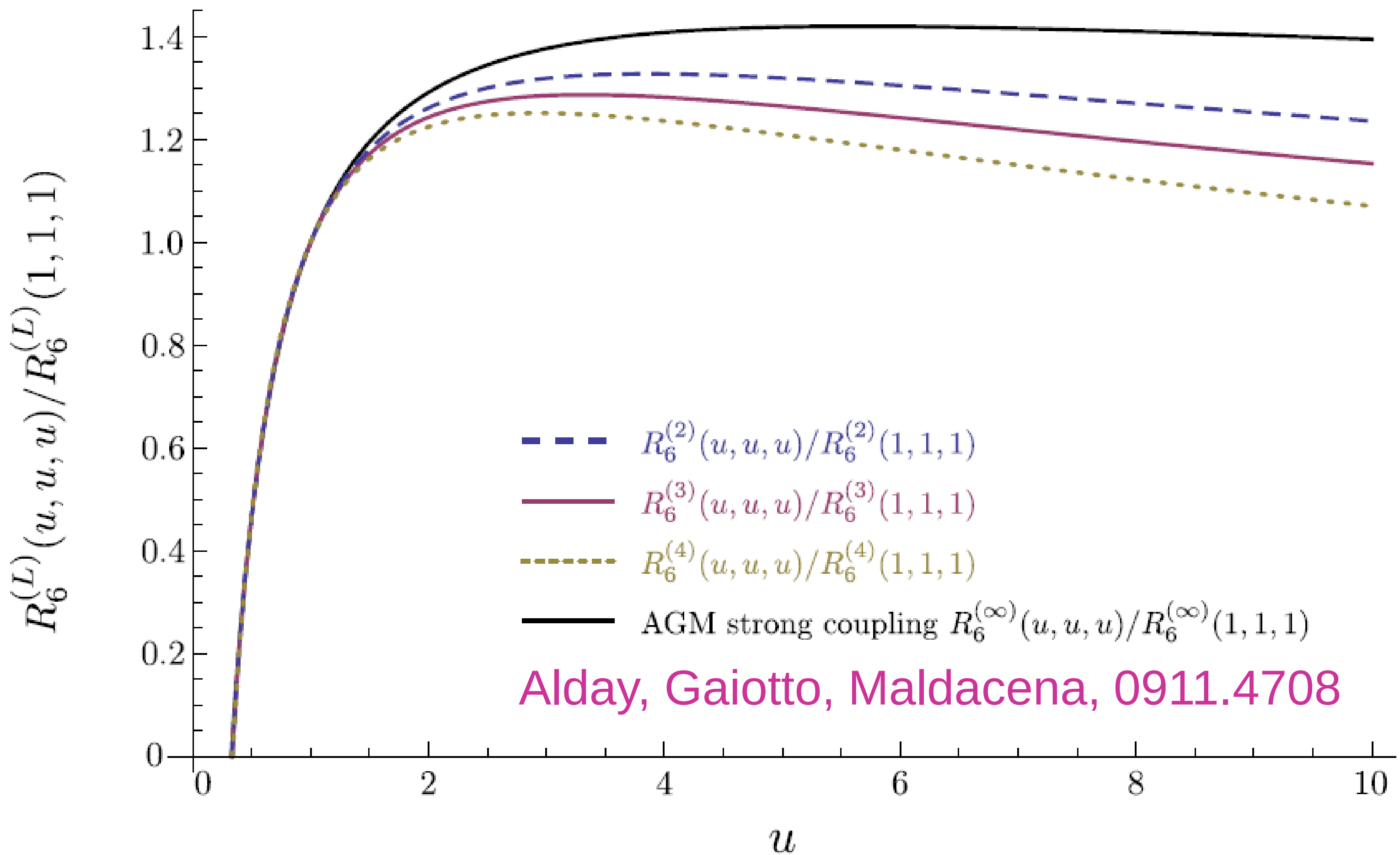
L	$\gamma_K^{(L)}/\gamma_K^{(L-1)}$	$\bar{R}_6^{(L)}(1,1,1)$	$\overline{\ln \mathcal{W}}_{\rm hex}^{(L)}(\frac{3}{4},\frac{3}{4},\frac{3}{4})$	$\overline{\ln \mathcal{W}}_{\rm hex}^{(L)}(\frac{1}{4},\frac{1}{4},\frac{1}{4})$
2	-1.6449340	∞	-2.7697175	-2.8015275
3	-3.6188549	-7.0040885	-5.0036164	-5.1380714
4	-4.9211827	-6.5880519	-5.8860842	-6.0359857
5	-5.6547494	-6.7092373	-6.3453695	-6.4658887
6	-6.0801089	–	–	–
7	-6.3589220	–	–	–
8	-6.5608621	–	–	–

↓ -8

On $(u,u,1)$, everything collapses to HPLs of u



Rescaled $R_6^{(L)}(u,u,u)$ and strong coupling



(u,u,u) → cyclotomic polylogs (weak coupling)
$\arccos^2(1/4/u)$ (strong coupling)

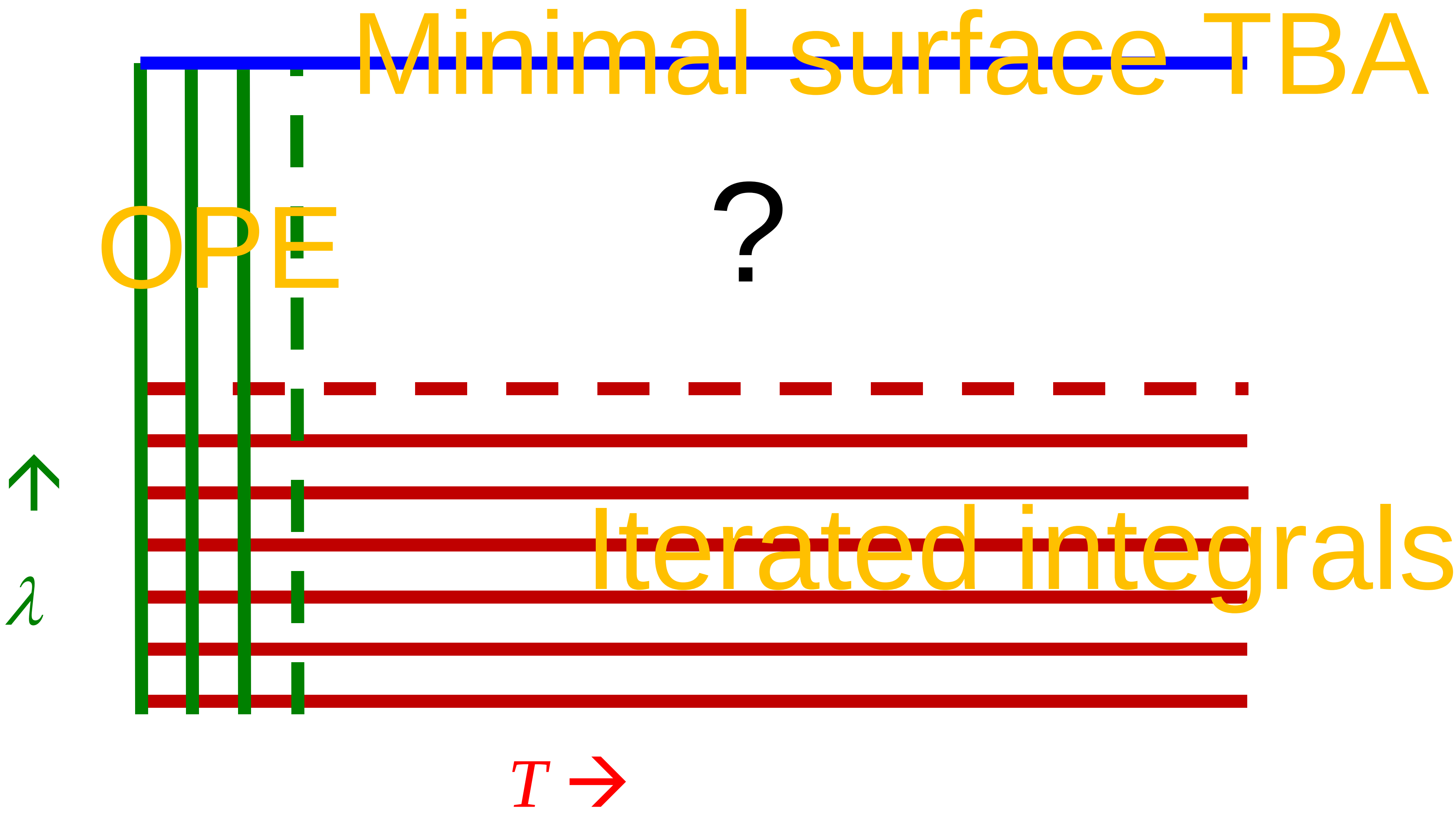
Beyond 6 gluons

- Cluster algebras provide strong clues to “the right functions” at least for MHV, NMHV

Golden, Goncharov, Paulos, Spradlin, Volovich, Vergu, 1305.1617, 1401.6446, 1411.3289

- E.g. symbol of 3-loop MHV 7-point amplitude
  Drummond, Papathanasiou, Spradlin 1412.3763
- Can turn such symbols into functions using same ideas discussed here
- Eventually elliptic functions will arise…

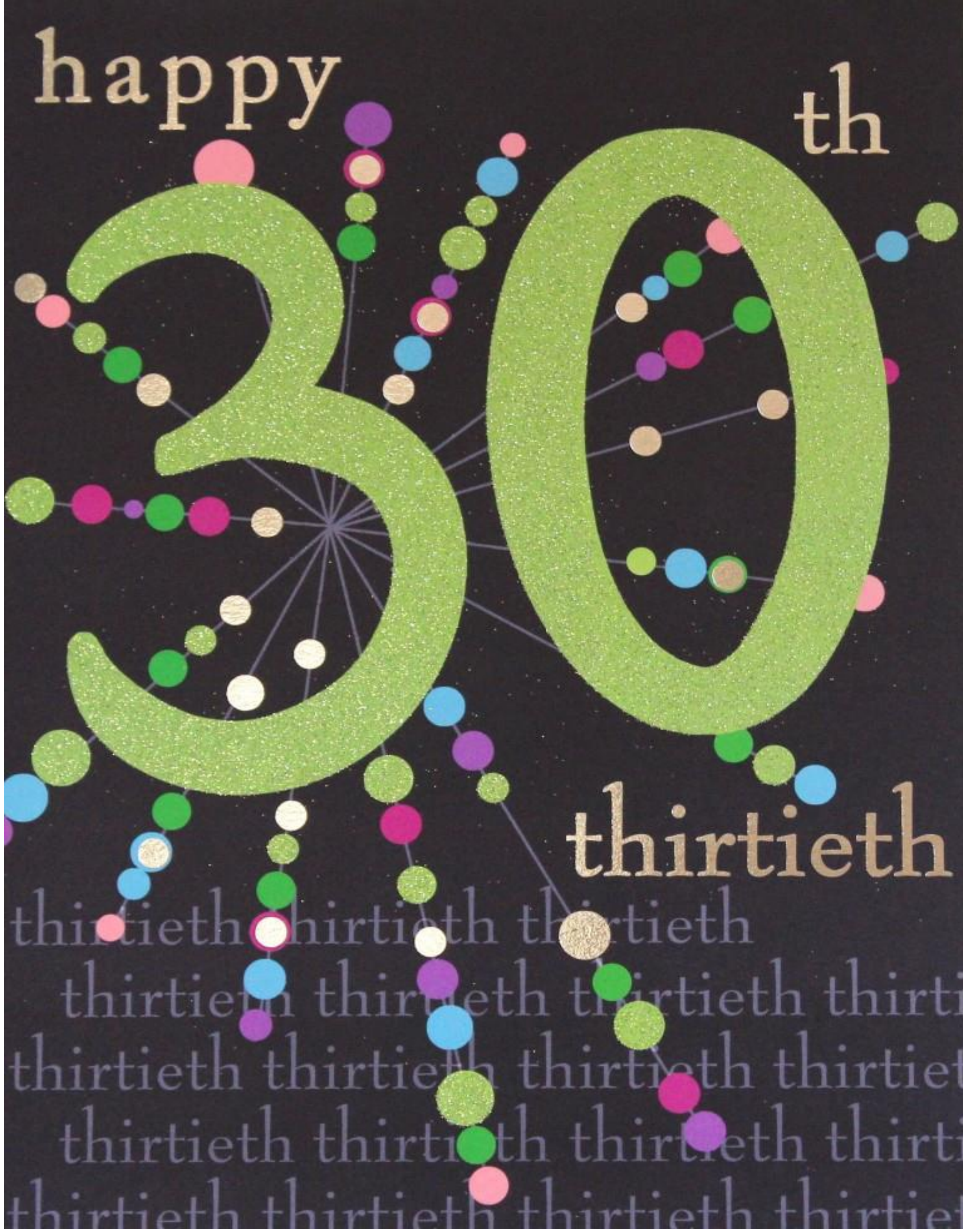
The big picture



Conclusions

- In planar N=4 SYM,
  MHV amplitudes = bosonic Wilson loops
  can be bootstrapped to high loop order
- Hexagon functions → 6 gluon amplitudes for all kinematics – through 5 loops so far
- Multiple cross checks and/or insights from studying OPE limits, self-crossing limits, ...
- Important avenue towards solving a 4-d QFT at finite coupling for generic kinematics

MHV



Iterative construction of hexagon functions

$$\frac{\partial F}{\partial u}\bigg|_{v,w} = \frac{F^u}{u} - \frac{F^{1-u}}{1-u} + \frac{1-u-v-w}{u\sqrt{\Delta}}F^{y_u} + \frac{1-u-v+w}{(1-u)\sqrt{\Delta}}F^{y_v} + \frac{1-u+v-w}{(1-u)\sqrt{\Delta}}F^{y_w}$$

- F weight n, from F^x weight n-1 (already classified)
- Just need to impose: 1. integrability $\frac{\partial^2 F}{\partial u_i \partial u_j} = \frac{\partial^2 F}{\partial u_j \partial u_i}, \qquad i \neq j$

$$\begin{aligned}
F^{u,v} &= F^{v,u} - F^{y_u,y_v} + F^{y_v,y_u}, \\
F^{v,w} &= F^{w,v} - F^{y_v,y_w} + F^{y_w,y_v}, \\
F^{w,u} &= F^{u,w} - F^{y_w,y_u} + F^{y_u,y_w}, \\
F^{1-u,1-v} &= F^{1-v,1-u} + F^{y_u,y_v} - F^{y_u,y_w} - F^{y_v,y_u} + F^{y_v,y_w} + F^{y_w,y_u} - F^{y_w,y_v} \\
F^{1-v,1-w} &= F^{1-w,1-v} + F^{y_v,y_w} - F^{y_v,y_u} - F^{y_w,y_v} + F^{y_w,y_u} + F^{y_u,y_v} - F^{y_u,y_w} \\
F^{1-w,1-u} &= F^{1-u,1-w} + F^{y_w,y_u} - F^{y_w,y_v} - F^{y_u,y_w} + F^{y_u,y_v} + F^{y_v,y_w} - F^{y_v,y_u} \\
F^{u,1-v} &= F^{1-v,u} + F^{y_u,y_w} - F^{y_w,y_u}, \\
F^{v,1-w} &= F^{1-w,v} + F^{y_v,y_u} - F^{y_u,y_v}, \\
F^{w,1-u} &= F^{1-u,w} + F^{y_w,y_v} - F^{y_v,y_w}, \\
F^{u,1-w} &= F^{1-w,u} + F^{y_u,y_v} - F^{y_v,y_u}, \\
F^{v,1-u} &= F^{1-u,v} + F^{y_v,y_w} - F^{y_w,y_v}, \\
F^{w,1-v} &= F^{1-v,w} + F^{y_w,y_u} - F^{y_u,y_w},
\end{aligned}$$

$$\begin{aligned}
F^{u,y_u} &= F^{y_u,u}, \\
F^{v,y_v} &= F^{y_v,v}, \\
F^{w,y_w} &= F^{y_w,w}, \\
F^{u,y_w} &= F^{w,y_u} - F^{y_u,w} + F^{y_w,u}, \\
F^{v,y_u} &= F^{u,y_v} - F^{y_v,u} + F^{y_u,v}, \\
F^{w,y_v} &= F^{v,y_w} - F^{y_w,v} + F^{y_v,w}, \\
F^{1-v,y_v} &= F^{y_v,1-v} - F^{y_u,1-u} + F^{1-u,y_u} + F^{y_u,w} - F^{w,y_u} - F^{y_w,v} + F^{v,y_w} \\
F^{1-w,y_w} &= F^{y_w,1-w} - F^{y_v,1-v} + F^{1-v,y_v} + F^{y_v,u} - F^{u,y_v} - F^{y_u,w} + F^{w,y_u} \\
F^{1-u,y_u} &= F^{y_u,1-u} - F^{y_w,1-w} + F^{1-w,y_w} + F^{y_w,v} - F^{v,y_w} - F^{y_v,u} + F^{u,y_v} \\
F^{1-u,y_v} &= F^{y_v,1-u} + F^{y_v,w} - F^{w,y_v}, \\
F^{1-v,y_w} &= F^{y_w,1-v} + F^{y_w,u} - F^{u,y_w}, \\
F^{1-w,y_u} &= F^{y_u,1-w} + F^{y_u,v} - F^{v,y_u}, \\
F^{1-u,y_w} &= F^{y_w,1-u} + F^{y_w,v} - F^{v,y_w}, \\
F^{1-v,y_u} &= F^{y_u,1-v} + F^{y_u,w} - F^{w,y_u}, \\
F^{1-w,y_v} &= F^{y_v,1-w} + F^{y_v,u} - F^{u,y_v}.
\end{aligned}$$

- 2. No bad branch cuts: $F^{1-u_i}(y_i = 1, y_j, y_k) = 0$

Multiple zeta values at $(u,v,w) = (1,1,1)$

$$R_6^{(2)}(1,1,1) = -(\zeta_2)^2 = -\frac{5}{2}\zeta_4$$

$$R_6^{(3)}(1,1,1) = \frac{413}{24}\zeta_6 + (\zeta_3)^2$$

First irreducible MZV

$$R_6^{(4)}(1,1,1) = -\frac{471}{4}\zeta_8 - \frac{3}{2}\zeta_2(\zeta_3)^2 - \frac{5}{2}\zeta_3\zeta_5 + \frac{3}{2}\zeta_{5,3}$$

$$R_6^{(5)}(1,1,1) = \frac{8389}{10}\zeta_{10} + 12\zeta_2\zeta_3\zeta_5 + 17\zeta_4(\zeta_3)^2 - \frac{63}{2}\zeta_3\zeta_7 - \frac{111}{8}(\zeta_5)^2 - \frac{3}{2}\zeta_2\zeta_{5,3} - 6\zeta_{7,3}$$

On the line $(u,u,1)$, everything collapses to HPLs of u.
In a linear representation,
and a very compressed notation,

$$H_1^u H_{2,1}^u = H_1^u H_{0,1,1}^u = 3H_{0,1,1,1}^u + H_{1,0,1,1}^u \to 3h_7^{[4]} + h_{11}^{[4]}$$

2 and 3 loop answers:

$$\begin{aligned}
R_6^{(2)}(u,u,1) &= h_1^{[4]} - h_3^{[4]} + h_9^{[4]} - h_{11}^{[4]} - \frac{5}{2}\zeta_4\,, \\
R_6^{(3)}(u,u,1) &= -3h_1^{[6]} + 5h_3^{[6]} + \frac{3}{2}h_5^{[6]} - \frac{9}{2}h_7^{[6]} - \frac{1}{2}h_9^{[6]} - \frac{3}{2}h_{11}^{[6]} - h_{13}^{[6]} - \frac{3}{2}h_{17}^{[6]} \\
&\quad + \frac{3}{2}h_{19}^{[6]} - \frac{1}{2}h_{21}^{[6]} - \frac{3}{2}h_{23}^{[6]} - 3h_{33}^{[6]} + 5h_{35}^{[6]} + \frac{3}{2}h_{37}^{[6]} - \frac{9}{2}h_{39}^{[6]} \\
&\quad - \frac{1}{2}h_{41}^{[6]} - \frac{3}{2}h_{43}^{[6]} - h_{45}^{[6]} - \frac{3}{2}h_{49}^{[6]} + \frac{3}{2}h_{51}^{[6]} - \frac{1}{2}h_{53}^{[6]} - \frac{3}{2}h_{55}^{[6]} \\
&\quad + \zeta_2\left[-h_1^{[4]} + 3h_3^{[4]} + 2h_5^{[4]} - h_9^{[4]} + 3h_{11}^{[4]} + 2h_{13}^{[4]}\right] \\
&\quad + \zeta_4\left[-2h_1^{[2]} - 2h_3^{[2]}\right] + \zeta_3^2 + \frac{413}{24}\zeta_6\,,
\end{aligned}$$

4 loop answer →

$$\begin{aligned}
R_6^{(4)}(u,u,1) = {} & 15h_1^{[8]} - 41h_3^{[8]} - \frac{31}{2}h_5^{[8]} + \frac{105}{2}h_7^{[8]} - \frac{7}{2}h_9^{[8]} + \frac{53}{2}h_{11}^{[8]} + 12h_{13}^{[8]} - 42h_{15}^{[8]} \\
& + \frac{5}{2}h_{17}^{[8]} + \frac{11}{2}h_{19}^{[8]} + \frac{9}{2}h_{21}^{[8]} - \frac{41}{2}h_{23}^{[8]} + h_{25}^{[8]} - 13h_{27}^{[8]} - 7h_{29}^{[8]} - 5h_{31}^{[8]} \\
& + 6h_{33}^{[8]} - 11h_{35}^{[8]} - 3h_{37}^{[8]} + 3h_{39}^{[8]} - 4h_{43}^{[8]} - 4h_{45}^{[8]} - 11h_{47}^{[8]} + \frac{3}{2}h_{49}^{[8]} - \frac{3}{2}h_{51}^{[8]} \\
& - 3h_{53}^{[8]} - 5h_{55}^{[8]} + \frac{3}{2}h_{57}^{[8]} - \frac{3}{2}h_{59}^{[8]} + 9h_{65}^{[8]} - 25h_{67}^{[8]} - 9h_{69}^{[8]} + 27h_{71}^{[8]} - 2h_{73}^{[8]} \\
& + 9h_{75}^{[8]} + 2h_{77}^{[8]} - 23h_{79}^{[8]} + 2h_{81}^{[8]} - h_{85}^{[8]} - 8h_{87}^{[8]} + 2h_{89}^{[8]} - 3h_{91}^{[8]} + \frac{5}{2}h_{97}^{[8]} \\
& - \frac{7}{2}h_{99}^{[8]} - \frac{1}{2}h_{101}^{[8]} + \frac{5}{2}h_{103}^{[8]} + \frac{1}{2}h_{105}^{[8]} + \frac{1}{2}h_{107}^{[8]} + \frac{1}{2}h_{109}^{[8]} - \frac{5}{2}h_{111}^{[8]} + 15h_{129}^{[8]} \\
& - 41h_{131}^{[8]} - \frac{31}{2}h_{133}^{[8]} + \frac{105}{2}h_{135}^{[8]} - \frac{7}{2}h_{137}^{[8]} + \frac{53}{2}h_{139}^{[8]} + 12h_{141}^{[8]} - 42h_{143}^{[8]} \\
& + \frac{5}{2}h_{145}^{[8]} + \frac{11}{2}h_{147}^{[8]} + \frac{9}{2}h_{149}^{[8]} - \frac{41}{2}h_{151}^{[8]} + h_{153}^{[8]} - 13h_{155}^{[8]} - 7h_{157}^{[8]} \\
& - 5h_{159}^{[8]} + 6h_{161}^{[8]} - 11h_{163}^{[8]} - 3h_{165}^{[8]} + 3h_{167}^{[8]} - 4h_{171}^{[8]} - 4h_{173}^{[8]} \\
& - 11h_{175}^{[8]} + \frac{3}{2}h_{177}^{[8]} - \frac{3}{2}h_{179}^{[8]} - 3h_{181}^{[8]} - 5h_{183}^{[8]} + \frac{3}{2}h_{185}^{[8]} - \frac{3}{2}h_{187}^{[8]} \\
& + 9h_{193}^{[8]} - 25h_{195}^{[8]} - 9h_{197}^{[8]} + 27h_{199}^{[8]} - 2h_{201}^{[8]} + 9h_{203}^{[8]} + 2h_{205}^{[8]} - 23h_{207}^{[8]} \\
& + 2h_{209}^{[8]} - h_{213}^{[8]} - 8h_{215}^{[8]} + 2h_{217}^{[8]} - 3h_{219}^{[8]} + \frac{5}{2}h_{225}^{[8]} - \frac{7}{2}h_{227}^{[8]} - \frac{1}{2}h_{229}^{[8]} \\
& + \frac{5}{2}h_{231}^{[8]} + \frac{1}{2}h_{233}^{[8]} + \frac{1}{2}h_{235}^{[8]} + \frac{1}{2}h_{237}^{[8]} - \frac{5}{2}h_{239}^{[8]} \\
& + \zeta_2\Big[2h_1^{[6]} - 14h_3^{[6]} - \frac{15}{2}h_5^{[6]} + \frac{37}{2}h_7^{[6]} - \frac{5}{2}h_9^{[6]} + \frac{25}{2}h_{11}^{[6]} + 7h_{13}^{[6]} - \frac{1}{2}h_{17}^{[6]} \\
& \qquad + \frac{5}{2}h_{19}^{[6]} + \frac{7}{2}h_{21}^{[6]} + \frac{9}{2}h_{23}^{[6]} - 3h_{25}^{[6]} + 3h_{27}^{[6]} + 2h_{33}^{[6]} - 14h_{35}^{[6]} - \frac{15}{2}h_{37}^{[6]} \\
& \qquad + \frac{37}{2}h_{39}^{[6]} - \frac{5}{2}h_{41}^{[6]} + \frac{25}{2}h_{43}^{[6]} + 7h_{45}^{[6]} - \frac{1}{2}h_{49}^{[6]} + \frac{5}{2}h_{51}^{[6]} + \frac{7}{2}h_{53}^{[6]} \\
& \qquad + \frac{9}{2}h_{55}^{[6]} - 3h_{57}^{[6]} + 3h_{59}^{[6]}\Big] \\
& + \zeta_4\left[\frac{15}{2}h_1^{[4]} - \frac{55}{2}h_3^{[4]} - \frac{41}{2}h_5^{[4]} + \frac{15}{2}h_9^{[4]} - \frac{55}{2}h_{11}^{[4]} - \frac{41}{2}h_{13}^{[4]}\right] \\
& + \left(\zeta_2\zeta_3 - \frac{5}{2}\zeta_5\right)\left[h_3^{[3]} + h_7^{[3]}\right] - \left(\zeta_3^2 - \frac{73}{4}\zeta_6\right)\left[h_1^{[2]} + h_3^{[2]}\right] \\
& - \frac{3}{2}\zeta_2\zeta_3^2 - \frac{5}{2}\zeta_3\zeta_5 - \frac{471}{4}\zeta_8 + \frac{3}{2}\zeta_{5,3}\,.
\end{aligned}$$

5 loop answer is several pages

MRK MHV Master formula

NLL: Fadin, Lipatov, 1111.0782;
Caron-Huot, 1309.6521

$$e^{R+i\pi\delta}|_{\rm MRK} = \cos\pi\omega_{ab} + i\,\frac{a}{2}\sum_{n=-\infty}^{\infty}(-1)^n\left(\frac{w}{w^*}\right)^{\frac{n}{2}}\int_{-\infty}^{+\infty}\frac{d\nu}{\nu^2+\frac{n^2}{4}}|w|^{2i\nu}\,\boxed{\Phi_{\rm Reg}(\nu,n)}$$

$$\times\left(-\frac{1}{1-u}\frac{|1+w|^2}{|w|}\right)^{\boxed{\omega(\nu,n)}}$$

$$w = -z, \quad w^* = -\bar{z}$$

MRK limits agree through 5 loops with all-orders predictions

Basso, Caron-Huot, Sever 1407.3766

- BFKL eigenvalue:

$$E^{(1)}(\nu,n),\ \ E^{(2)}(\nu,n),\ \ E^{(3)}(\nu,n),\ \ E^{(4)}(\nu,n)$$

LL, NLL, NNLL, NNNLL

- Impact factors:

$$\Phi^{(1)}_{\rm Reg}(\nu,n),\ \ \Phi^{(2)}_{\rm Reg}(\nu,n),\ \ \Phi^{(3)}_{\rm Reg}(\nu,n),\ \ \Phi^{(4)}_{\rm Reg}(\nu,n)$$

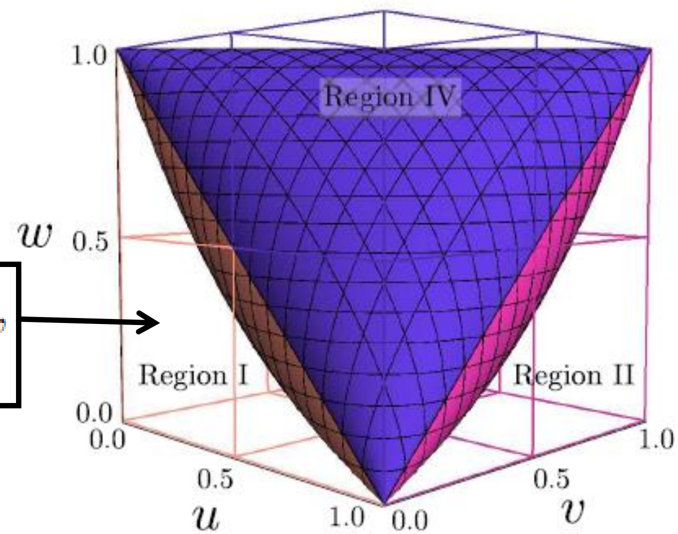
- All zeta-valued linear combinations of:

derivatives of $\ln\Gamma(1\pm i\nu+\frac{n}{2})$ $\quad \frac{i\nu}{\nu^2+\frac{n^2}{4}},\ \ \frac{n}{\nu^2+\frac{n^2}{4}}$

Hexagon functions are multiple polylogarithms in y_i

$$G(a_1,\ldots,a_n;z) = \int_0^z \frac{dt}{t-a_1} G(a_2,\ldots,a_n;t)$$

Region I : $\begin{cases} \Delta > 0\,, \quad 0 < u_i < 1\,, \quad \text{and} \quad u+v+w<1, \\ 0 < y_i < 1\,. \end{cases}$



$$\mathcal{G} = \Big\{G(\vec{w};y_u)|w_i \in \{0,1\}\Big\} \cup \Big\{G(\vec{w};y_v)\Big| w_i \in \Big\{0,1,\frac{1}{y_u}\Big\}\Big\} \cup \Big\{G(\vec{w};y_w)\Big| w_i \in \Big\{0,1,\frac{1}{y_u},\frac{1}{y_v},\frac{1}{y_u y_v}\Big\}\Big\}$$

- Useful for analytics and for numerics for $\Delta > 0$

  GiNaC implementation: Vollinga, Weinzierl, hep-th/0410259

A menagerie of functions

1. HPLs: One variable, symbol letters $\{u, 1-u\}$. Near-collinear limit, lines $(u,u,1)$, $(u,1,1)$
2. Cyclotomic Polylogarithms [Ablinger, Blumlein, Schneider, 1105.6063]: One variable, letters $\{y_u, 1+y_u, 1+y_u+y_u^2\}$. For line (u,u,u).
3. SVHPLs [F. Brown, 2004]: Two variables, letters $\{z, 1-z, \bar{z}, 1-\bar{z}\}$. First entry/single-valuedness constraint (real analytic function in z plane). Multi-Regge limit.
4. Full hexagon functions. Three variables, symbol letters $\{u, v, w, 1-u, 1-v, 1-w, y_u, y_v, y_w\}$, branch-cut condition