Geometry of non-planar amplitudes

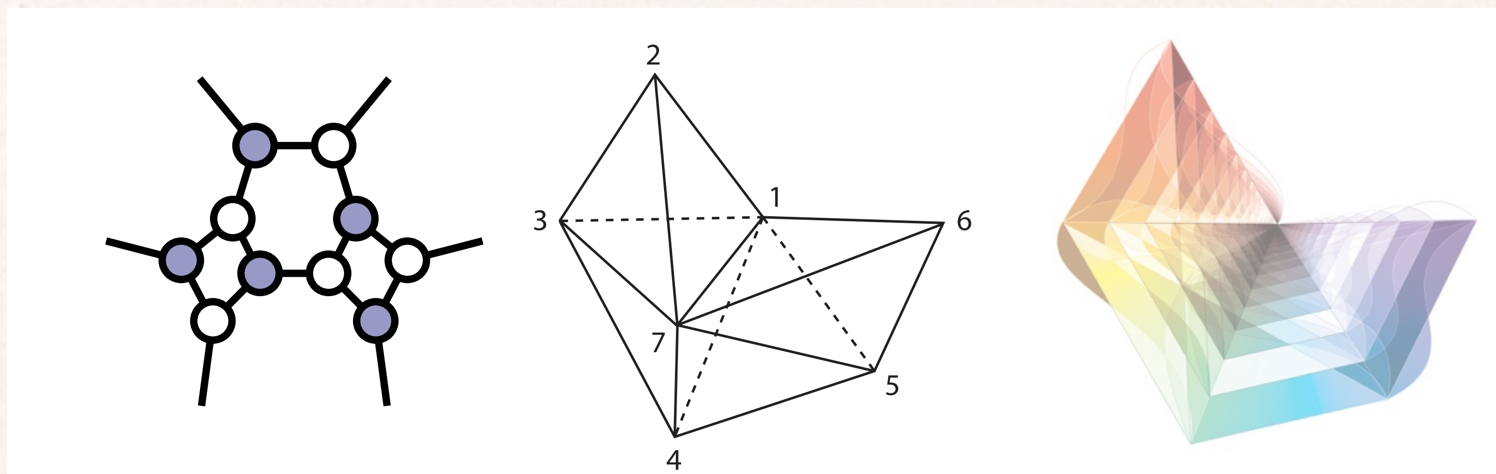
with
Nima Arkani-Hamed
Zvi Bern
Jacob Bourjaily
Freddy Cachazo
Enrico Herrmann
Andrew Hodges
Sean Litsey
Alexander Postnikov
James Stankowicz

Jaroslav Trnka (QMAP, UC Davis)

MHV @ 30, Fermilab, March 17, 2016

Goal

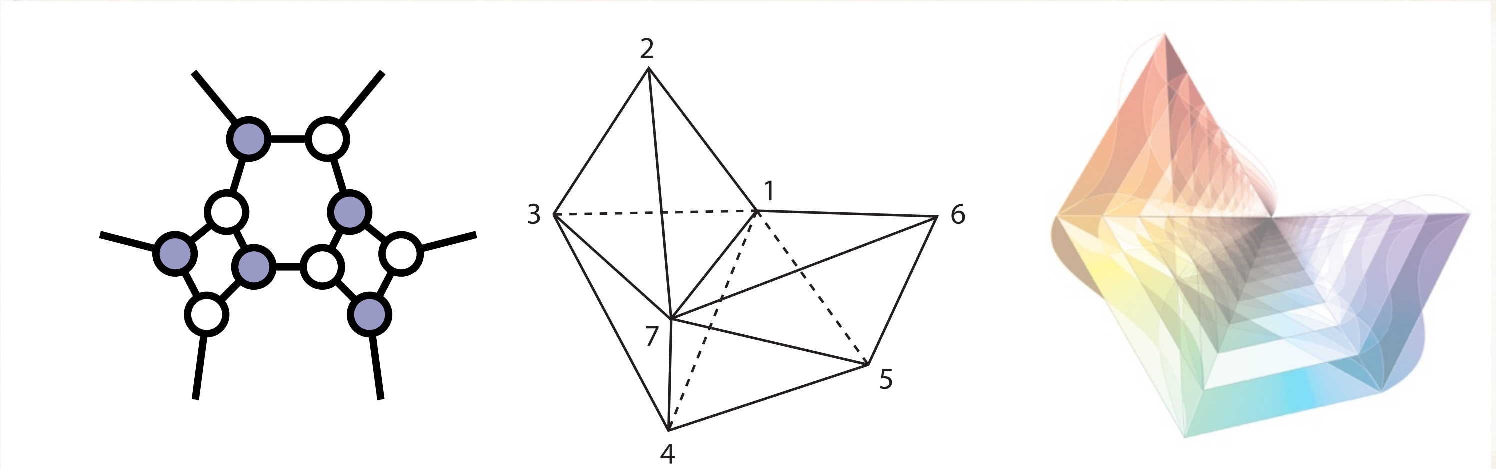
Mathematical structures in planar N=4 SYM



Other theories

Goal

Mathematical structures in planar N=4 SYM



Non-planar N=4 SYM

Plan of the talk

- Geometric picture for integrand in planar N=4 SYM
- Singularity structure of non-planar amplitudes
- Towards supergravity amplitudes

Hidden simplicity in amplitudes

- Once upon a time there was a MHV amplitude….

$$A = \frac{\langle 12 \rangle^4}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \ldots \langle n1 \rangle}$$

First evidence for simplicity in scattering amplitudes

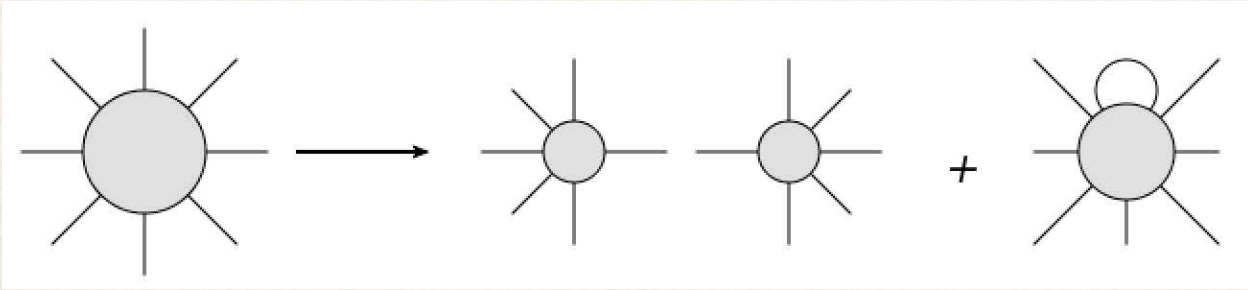
- Amplitudes are more than sums of Feynman diagrams

Singularities of amplitudes

- Scattering amplitudes of massless particles in D=4

$$\mathcal{A} = \sum_j \int d\mathcal{I}_j = \int d\mathcal{I}$$

- General idea: amplitudes are **fixed** from their singularities
- Locality: only $\frac{1}{P^2}$ present
- Unitarity: factorization on poles



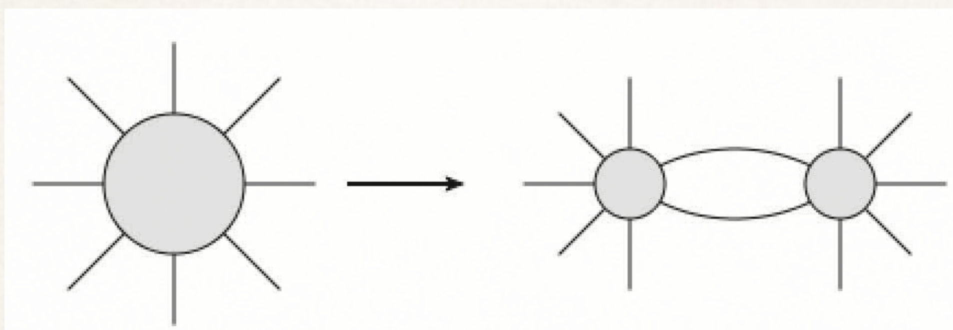
Integrand is an ideal object to construct/study

Unitarity methods

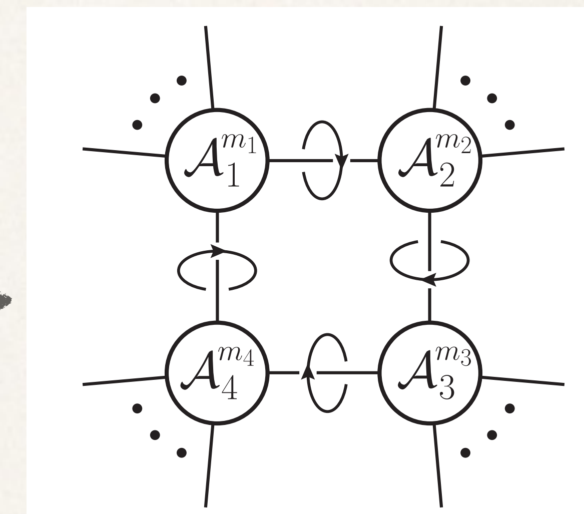


(Bern, Dixon, Kosower)

- Iterative use of the unitary cut

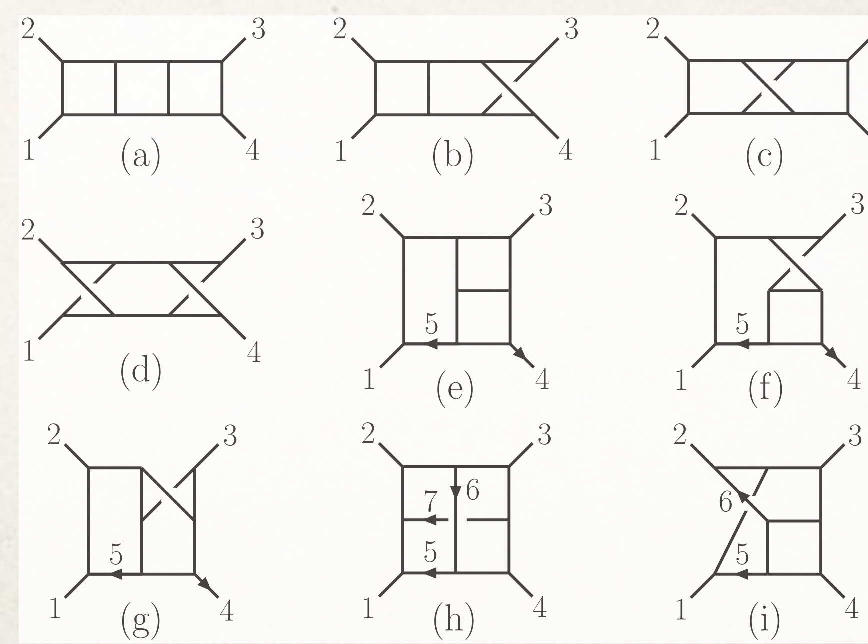


Generalized unitarity



(Britto, Cachazo, Feng)

- Generate basis of integrals, fixing coefficients from cuts



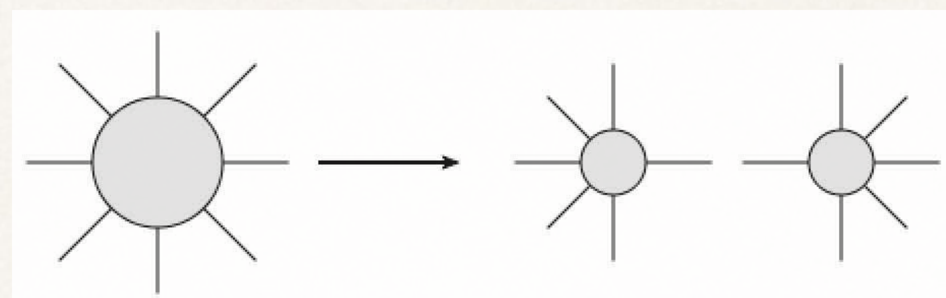
- Tremendous success in calculations in 1990-today

Blackhat: QCD background

BCFW recursion relations

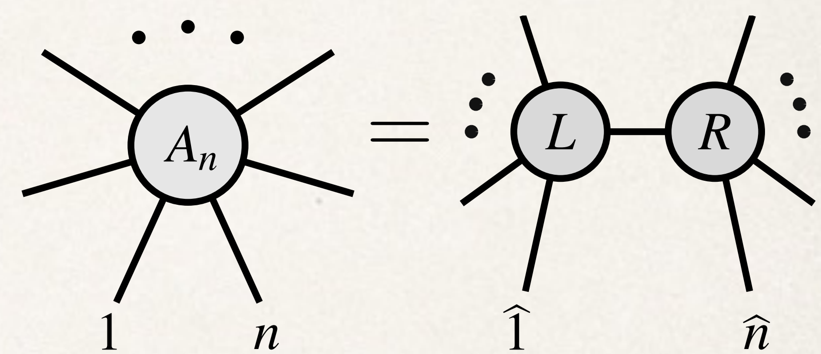
(Britto, Cachazo, Feng, Witten 2005)

- Large class of theories at tree-level
- Tree-level unitarity



- Shift momenta + Cauchy formula

$$p_1 \to p_1 + zq$$
$$p_2 \to p_2 - zq$$



- Very efficient method:

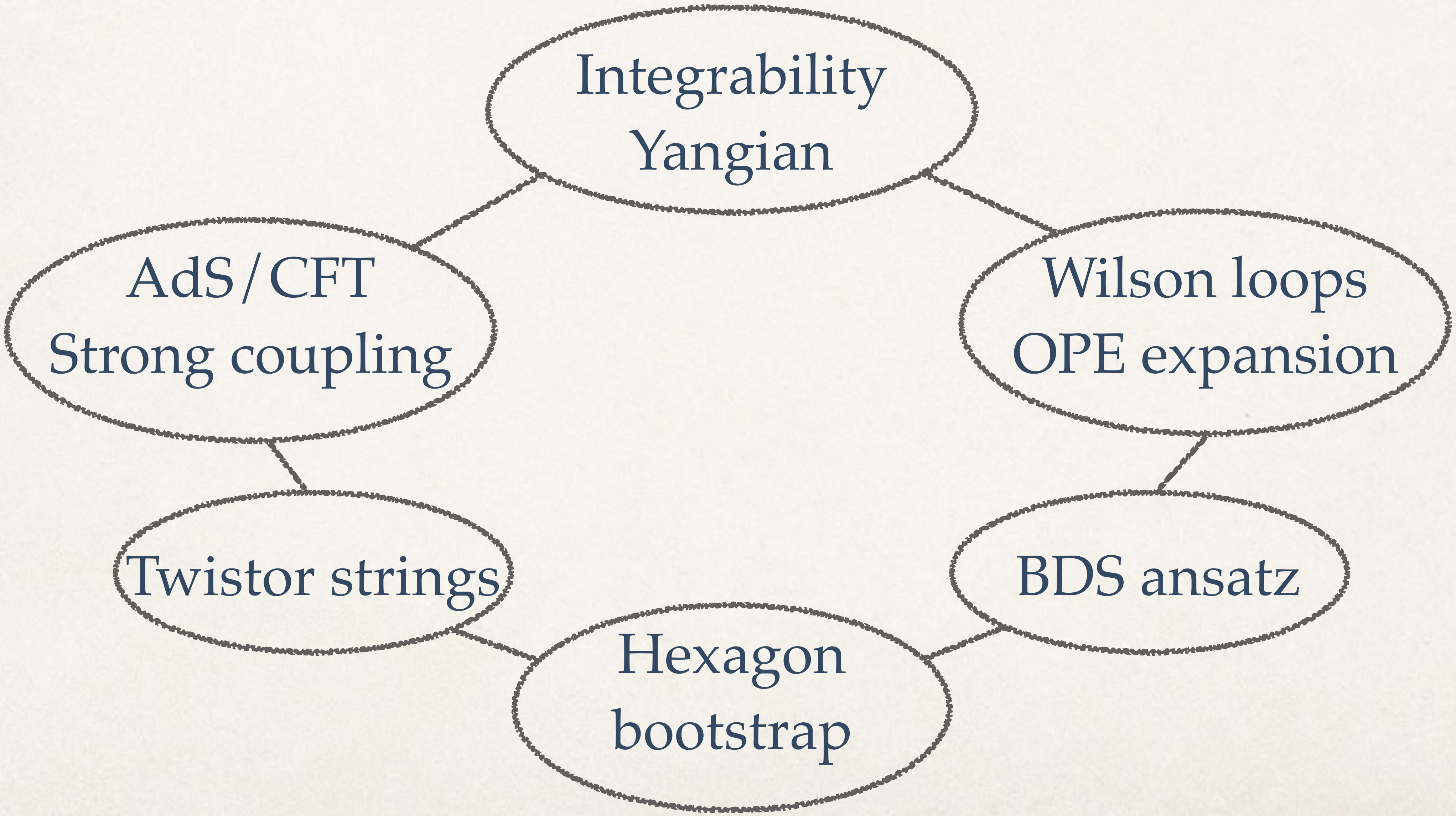
	$gg \to 4g$	$gg \to 5g$	$gg \to 6g$
Feynman diagrams	220	2485	34300
Recursion relations	3	6	20

Hydrogen atom of gauge theories

- N=4 Super Yang-Mills theory in the planar limit
- Great toy model for QCD
  - Tree-level amplitudes identical
  - Convergent perturbative series, no confinement
  - Hidden symmetries in the theory
- Past: new methods for amplitudes originated here

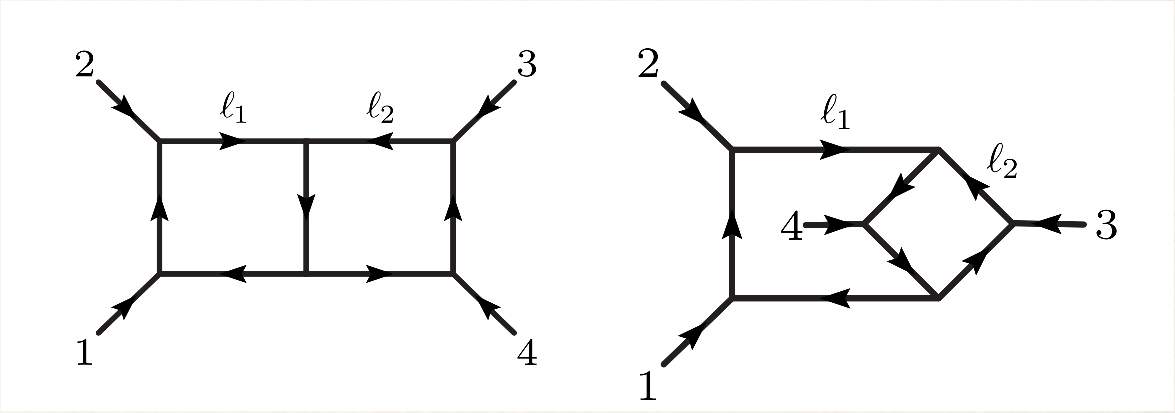
Planar N=4 SYM theory

- Useful playground for many theoretical ideas

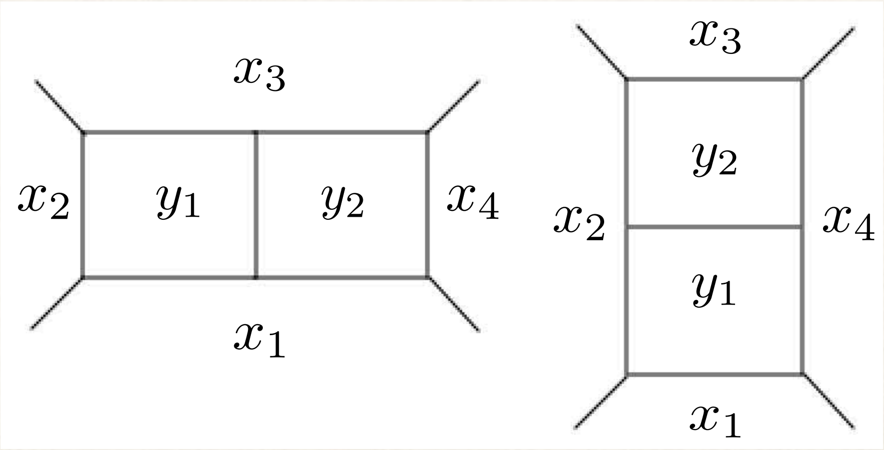


Dual variables

- Generally, each diagram has its own variables
  - No global loop momenta
  - Each diagram: its own labels



- Planar limit: dual variables $p_i = x_{i+1} - x_i$



$$k_1 = (x_1 - x_2) \quad k_2 = (x_2 - x_3)$$
$$\ell_1 = (x_3 - y_1) \quad \ell_2 = (y_2 - x_3)$$
etc

Global variables

Dual conformal symmetry

- Using these variables: define a single function

$$\mathcal{M} = \int d^4y_1 \ldots d^4y_L \, \mathcal{I}(x_i, y_j)$$

Integrand

Unique in planar N=4 SYM

- Tree-level amplitudes + integrand in planar N=4 SYM:

  Dual conformal symmetry (Drummond, Henn, Smirnov, Sokatchev 2007)

- Superconformal symmetry + Dual -> Yangian

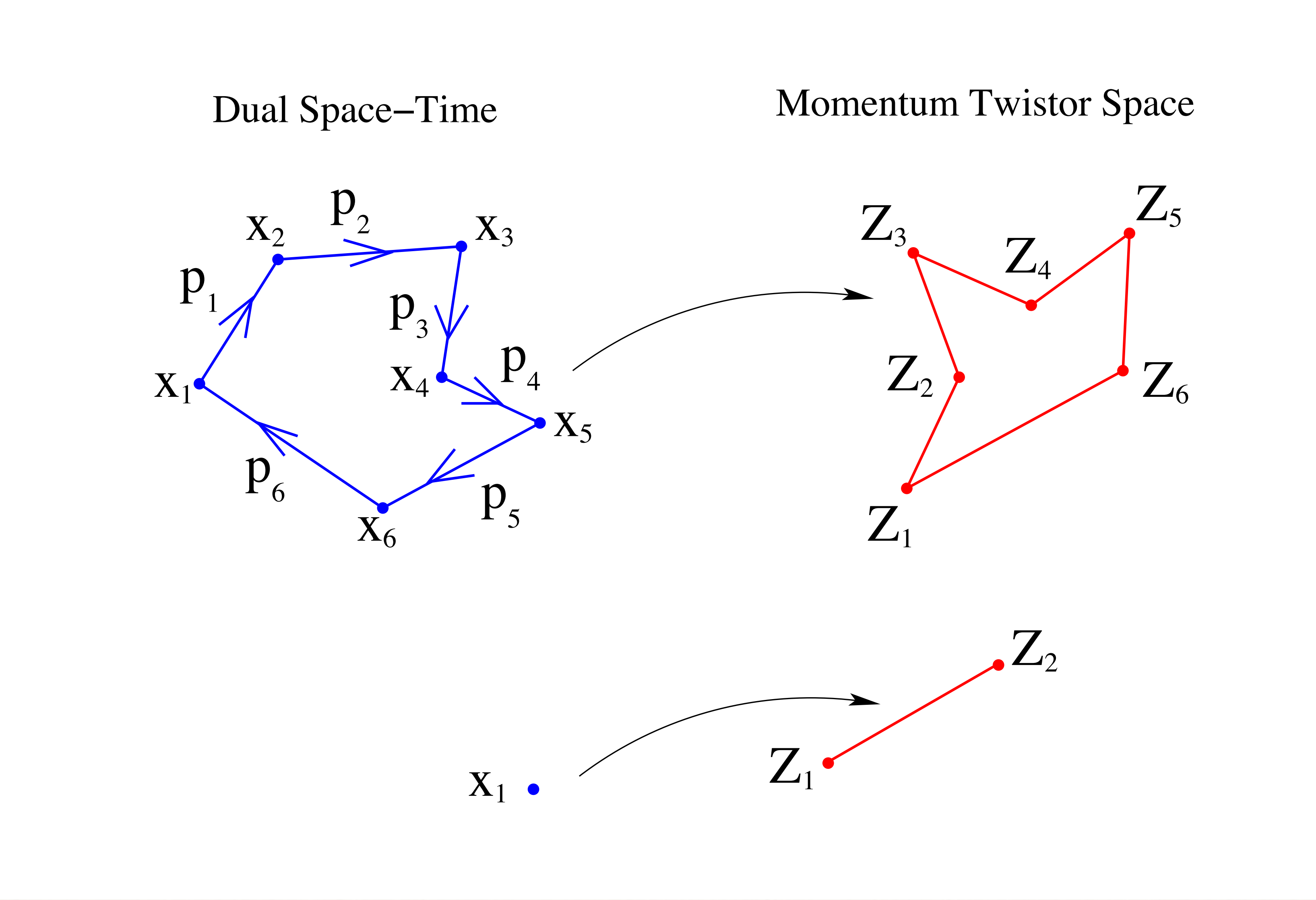
  (Drummond, Henn, Korchemsky, Sokatchev 2008) (Drummond, Henn, Plefka 2009)

Momentum twistors

(Hodges 2009)

- New variables: points in $\mathbb{P}^3$

$$Z = \begin{pmatrix} \lambda_a \\ x_{a\dot{a}}\lambda_a \end{pmatrix}$$



Dual conformal symmetry acts as SL(4) on Z

Momentum twistors

- Dual conformal invariants: $\langle 1234\rangle = \epsilon_{abcd} Z_1^a Z_2^b Z_3^c Z_4^d$

- Functions of momenta only: projective

box integral

$$\ell^2 = \frac{\langle AB41\rangle}{\langle AB\rangle\langle 41\rangle}$$

cross ratio

$$\frac{\langle ABd^2A\rangle\langle ABd^2B\rangle\langle 1234\rangle^2}{\langle AB12\rangle\langle AB23\rangle\langle AB34\rangle\langle AB41\rangle}$$

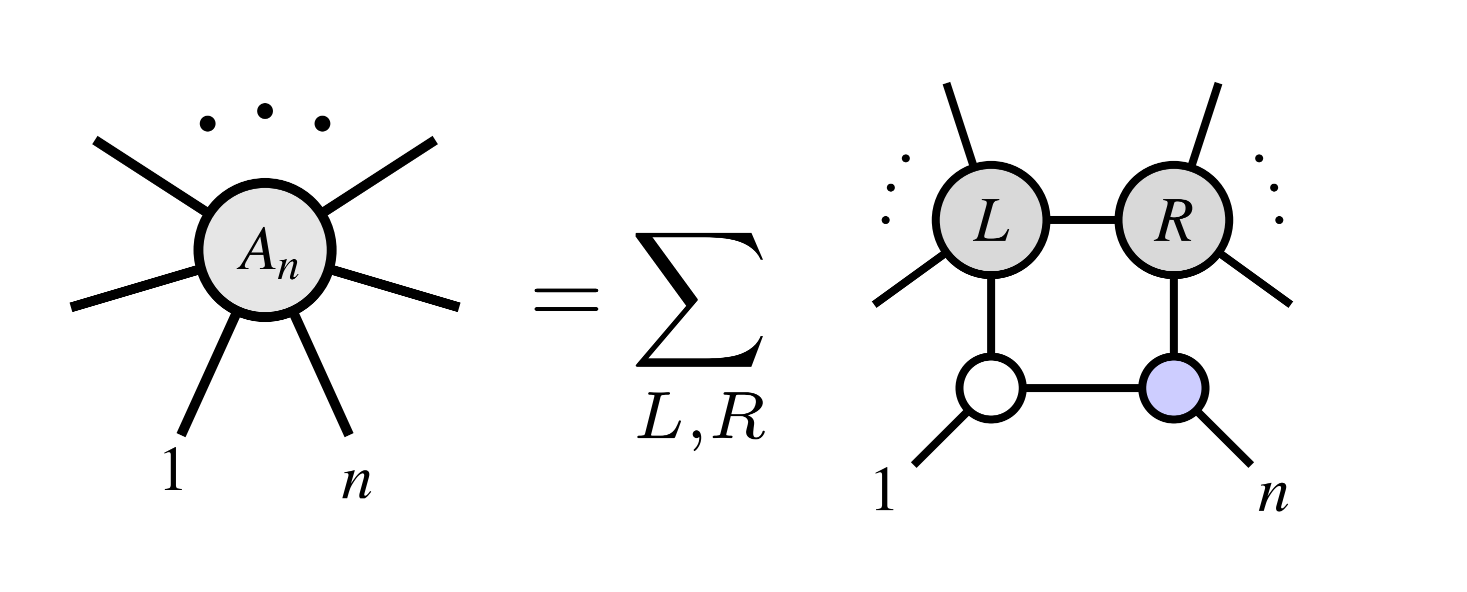
$$\frac{\langle 1234\rangle\langle 4561\rangle}{\langle 1245\rangle\langle 3461\rangle}$$

- Infinity twistor I^{ab} breaks dual conformal symmetry

$$\langle 12\rangle = \langle 12I\rangle = \epsilon_{abcd} Z_1^a Z_2^b I^{cd}$$

Manifest Yangian symmetry

- Terms in BCFW recursion: products of on-shell amplitudes



Tension between locality and symmetry

- Each term separately Yangian invariant

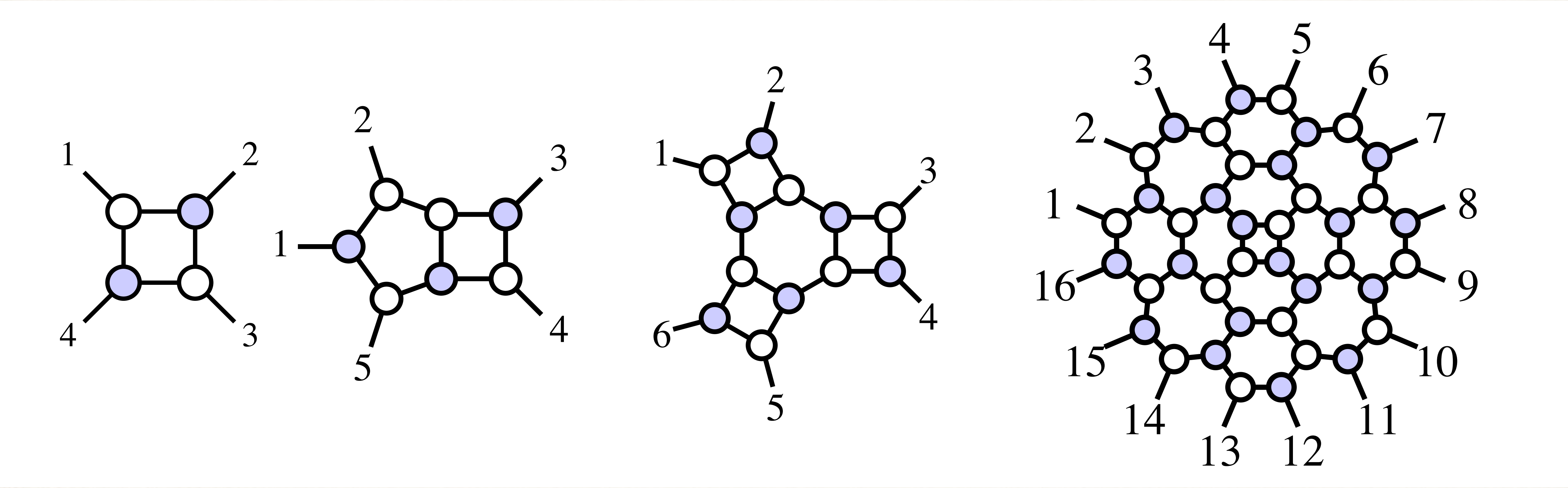
- Iterate until all vertices are 3pt: **on-shell diagrams**

On-shell diagrams

(Arkani-Hamed, Bourjaily, Cachazo, Goncharov, Postnikov, JT 2012)

On-shell diagrams

- Draw arbitrary graph with three point vertices

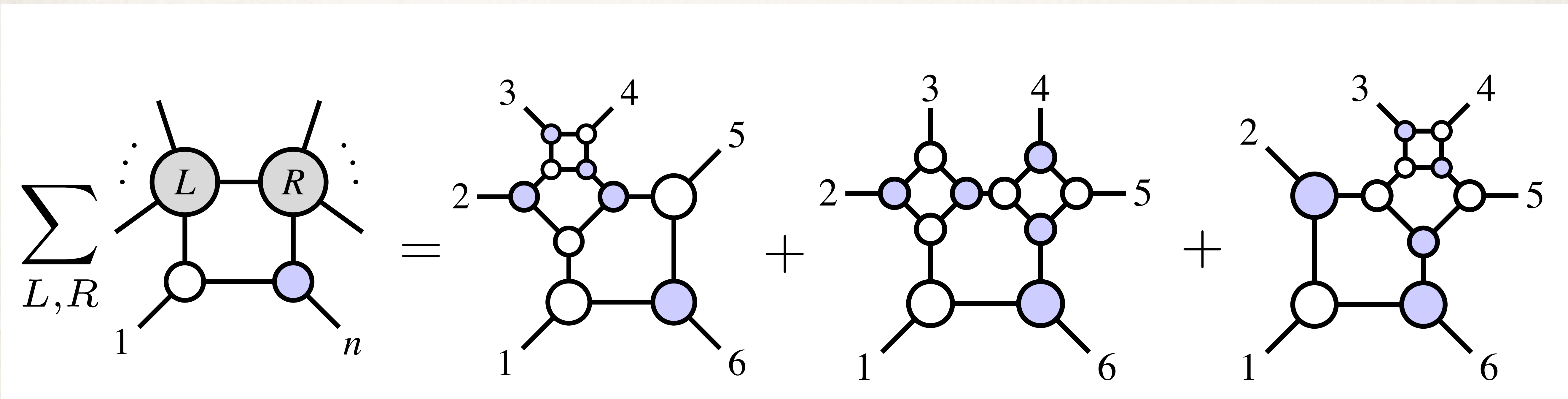


Products of three point amplitudes

$$\begin{cases} P > 4L & \text{Extra delta functions} \\ P = 4L & \text{Function of external data only} \\ P < 4L & \text{Unfixed parameters (forms)} \end{cases}$$

On-shell diagram expansion

- Example of 6pt amplitude



- Each diagram: on-shell, gauge invariant function

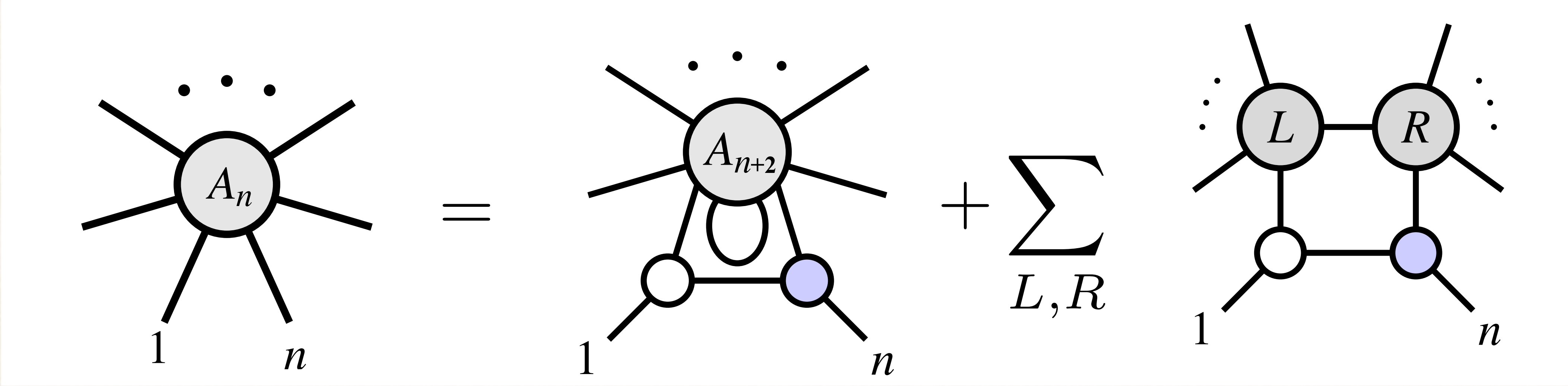
  Planar N=4: Yangian invariant $[12345]$

- Same pictures: cuts of the loop amplitudes with $\delta(P^2)$

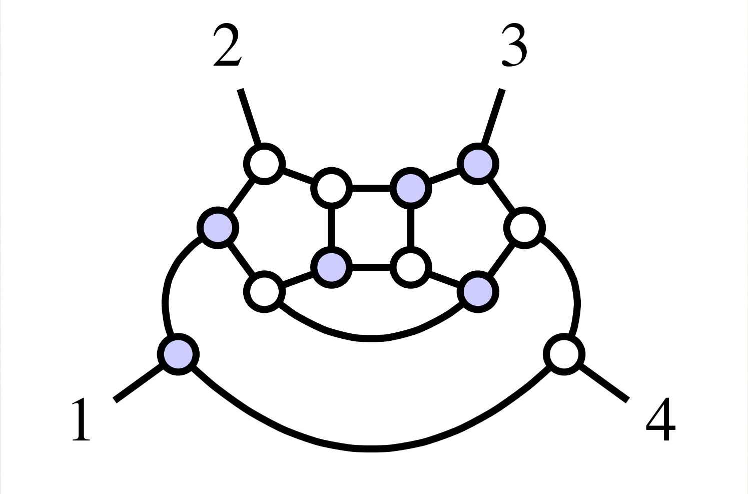
Loop recursion relations

(Arkani-Hamed, Bourjaily, Cachazo, Caron-Huot, JT 2010)

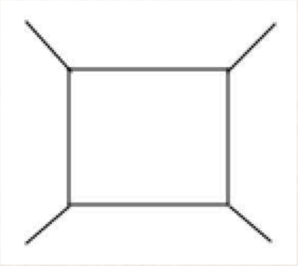
- Recursion relations for ℓ-loop integrand



- Example: 4pt 1-loop

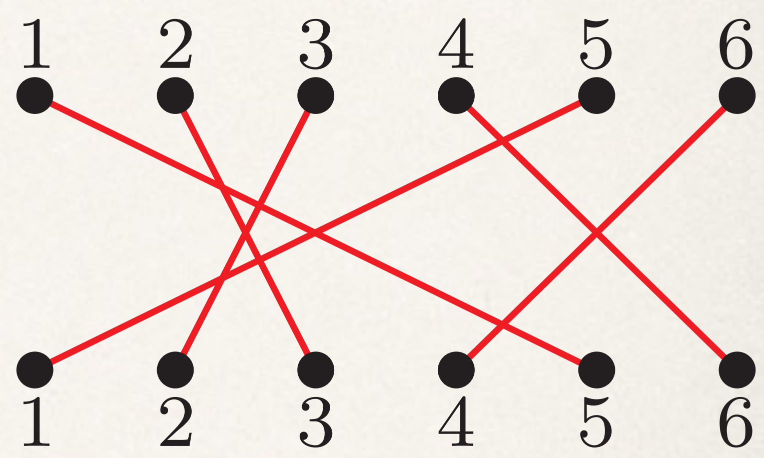


5-loop on-shell diagram = 1-loop off-shell box

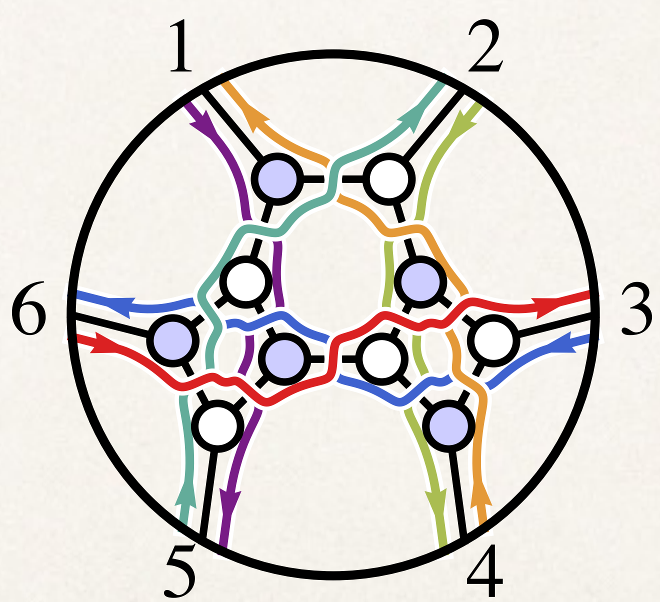


Permutations

- Represent graphically permutation



- Graph with only 3pt vertices

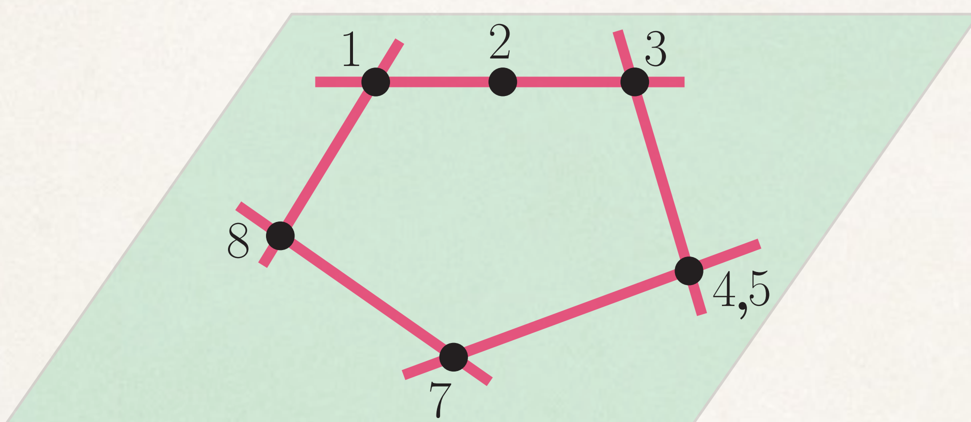


  - Turn right on blue
  - Turn left on white

Tree-level amplitudes: list of permutations

Positive Grassmannian

- Space of n points in k-dim projective space with linear dependencies between consecutive points



$$\leftrightarrow \quad \begin{pmatrix} * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \end{pmatrix}$$

- $(k \times n)$ real matrix with positive main $(k \times k)$ minors

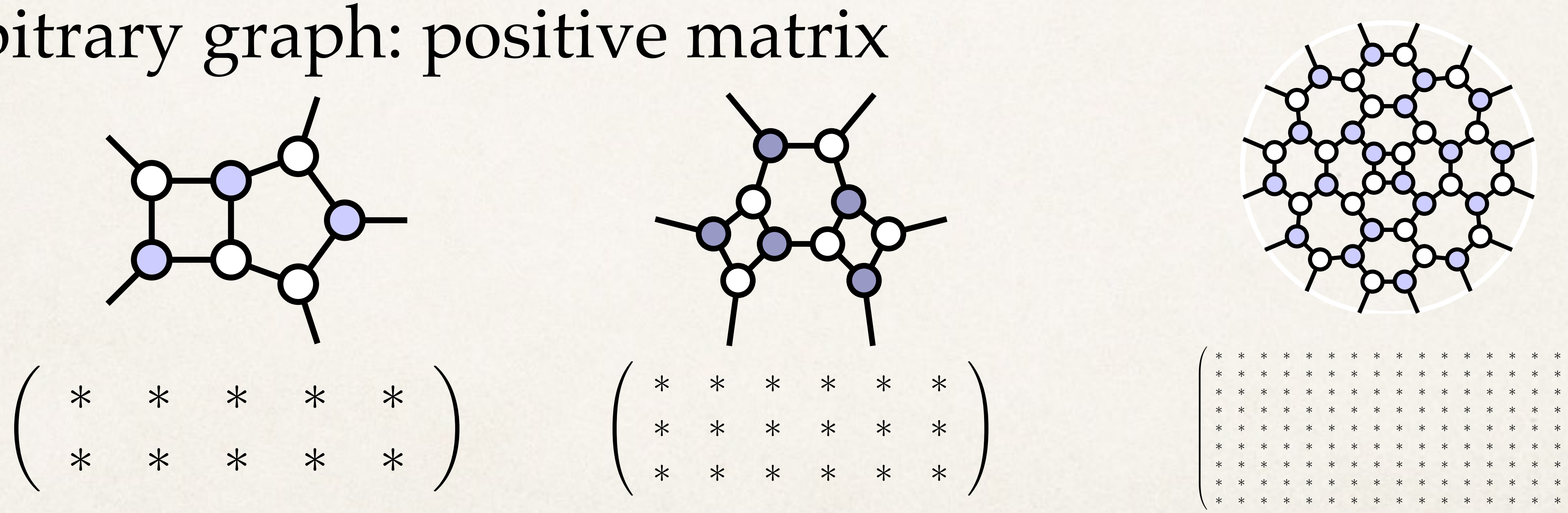
- How to construct this matrix? Using the same diagrams

Amalgamation procedure

- Construct big positive matrix from small ones

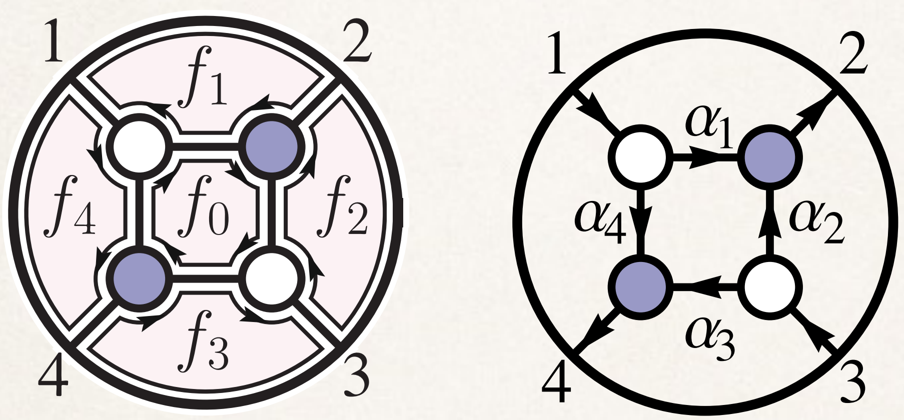
Gluing preserves positivity of minors

- Arbitrary graph: positive matrix



Connection to amplitudes

- Building positive matrix: face or edge variables



$$C = \begin{pmatrix} 1 & \alpha_1 & 0 & -\alpha_4 \\ 0 & \alpha_2 & 1 & \alpha_3 \end{pmatrix} \qquad \alpha_i \geq 0$$

- Same function as a product of 3pt amplitudes equal to

$$\Omega = \frac{d\alpha_1}{\alpha_1}\frac{d\alpha_2}{\alpha_2}\frac{d\alpha_3}{\alpha_3}\frac{d\alpha_4}{\alpha_4}\delta(C\cdot Z)$$

→ Solves for α_i in terms of $\lambda_i, \widetilde{\lambda}_i$ and gives $\delta(P)\delta(Q)$

Logarithmic singularities

- Amplitude = sum of on-shell diagrams

$$\Omega \sim \frac{d\alpha}{\alpha} \quad \text{near any pole} \quad \alpha = F(\ell_j, p_i)$$

- More than single poles: $\displaystyle \frac{dx\,dy}{xy(x+y)} \xrightarrow{x=0} \frac{dy}{y^2}$

- Logarithmic singularities specific for planar N=4 SYM

Generic QFT: $\Omega = F(\alpha)\,\delta(C\cdot Z)$

- Dlog form: close relation to maximal transcendentality

Geometric interpretation

- On-shell diagrams: regions (cells) in the Grassmannian
- Logarithmic form: “volume” of this region
- Amplitude: sum of on-shell diagrams

  Given by BCFW: unitarity
- Question: Is there a complete geometric definition?

Amplituhedron

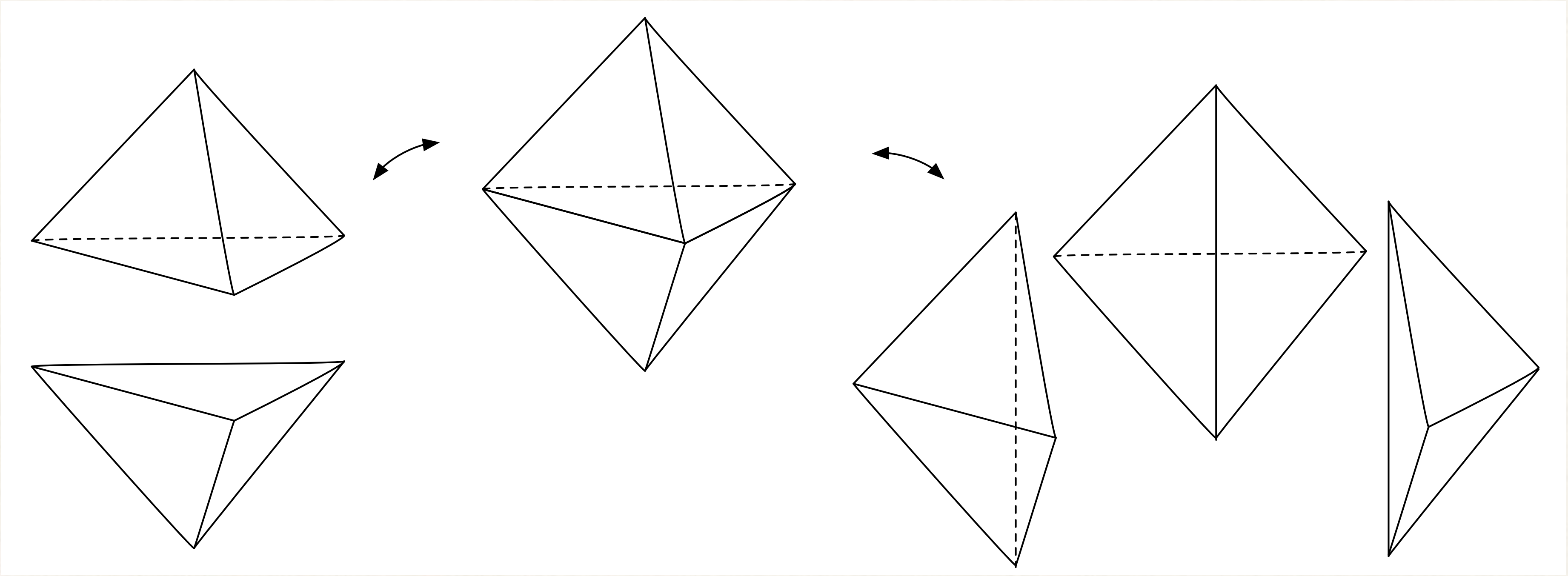
(Arkani-Hamed, JT 2013)

Volume of polyhedron

(Hodges 2009)

- Tree-level process: $gg \rightarrow 5g$ in momentum twistor space

- Comparison of two calculations of recursion relations

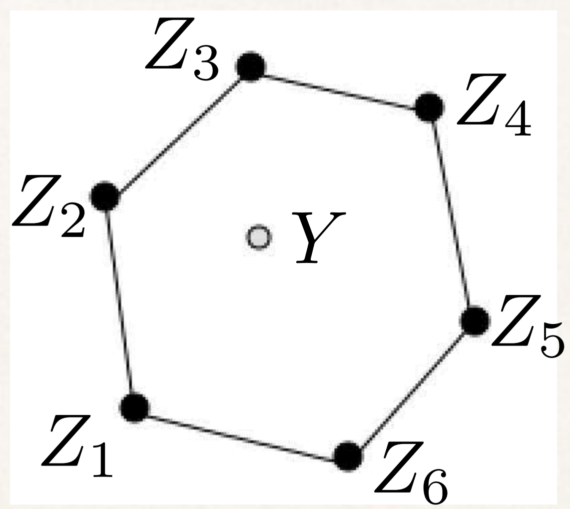


(Picture by Stavros Garoufalidis)

- Even simpler case: polygon

Point inside the polygon

- Consider a point inside a polygon in projective plane



$$Y = c_1 Z_1 + c_2 Z_2 + \ldots c_n Z_n$$

Space of all points inside convex polygon

Form with logarithmic singularities on boundaries

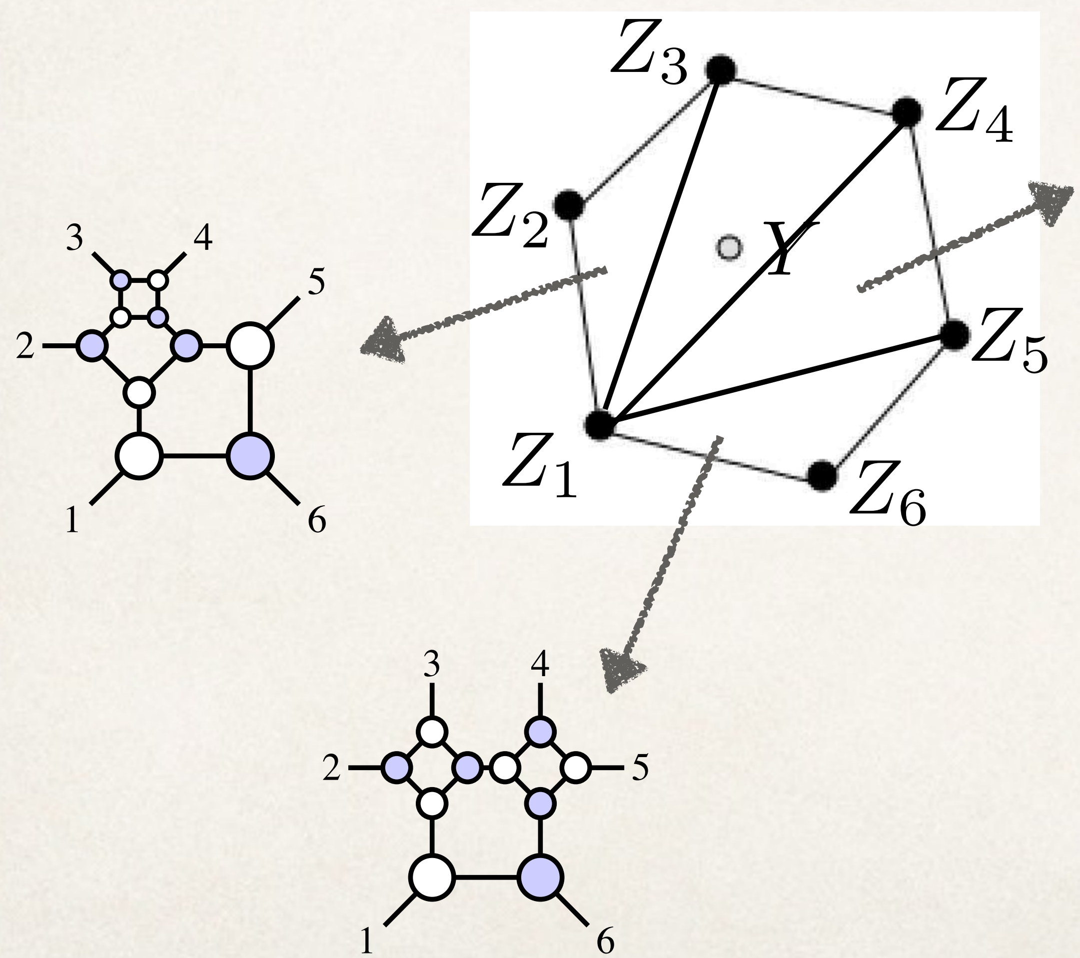
$$\Omega(Y, Z_i)$$

$$C = \begin{pmatrix} c_1 & c_2 & c_3 & \ldots & c_n \end{pmatrix} \in G_+(1,n)$$

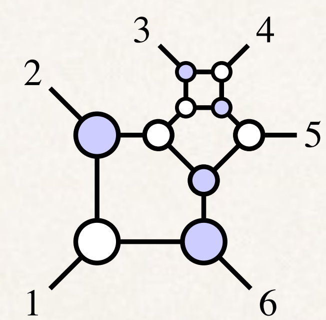
$$Z = \begin{pmatrix} \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\ Z_1 & Z_2 & Z_3 & \ldots & Z_n \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \end{pmatrix} \in M_+(3,n)$$

Triangulation

- BCFW using on-shell diagrams is a triangulation



$$C = \begin{pmatrix} 1 & 0 & 0 & c_4 & c_5 & 0 \end{pmatrix} \in G_+(1,6)$$

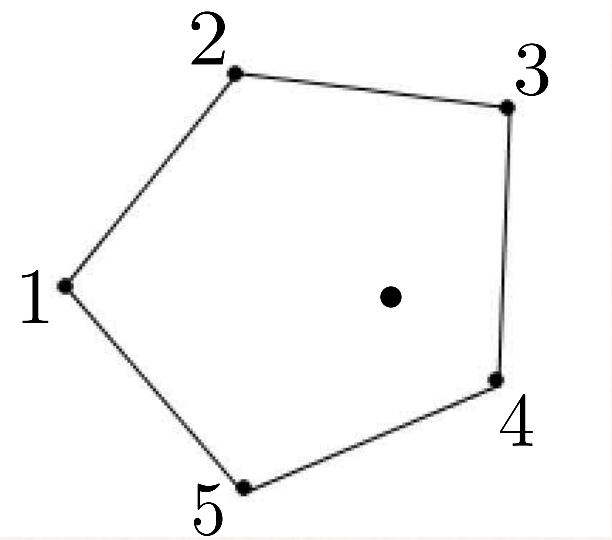


$$\Omega = \frac{dc_4}{c_4}\frac{dc_5}{c_5}$$

Supersymmetry
-> higher dimensional
bosonic space

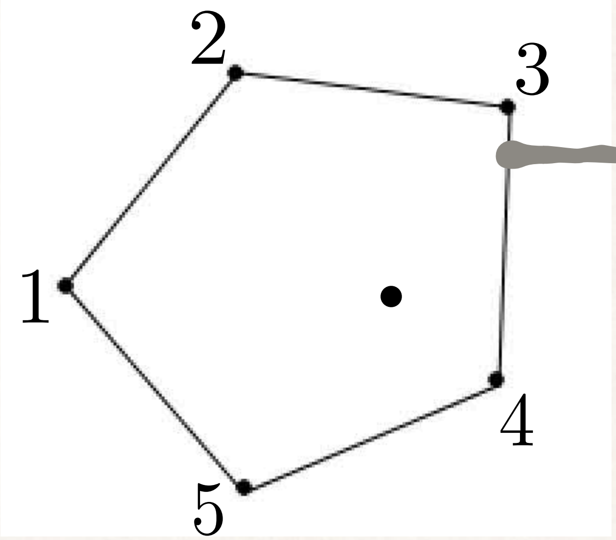
Road to Amplituhedron

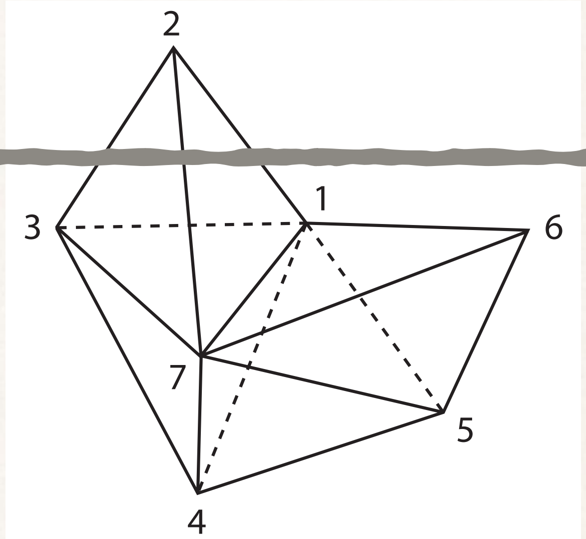
<u>Start:</u>
Point inside a convex polygon



Road to Amplituhedron

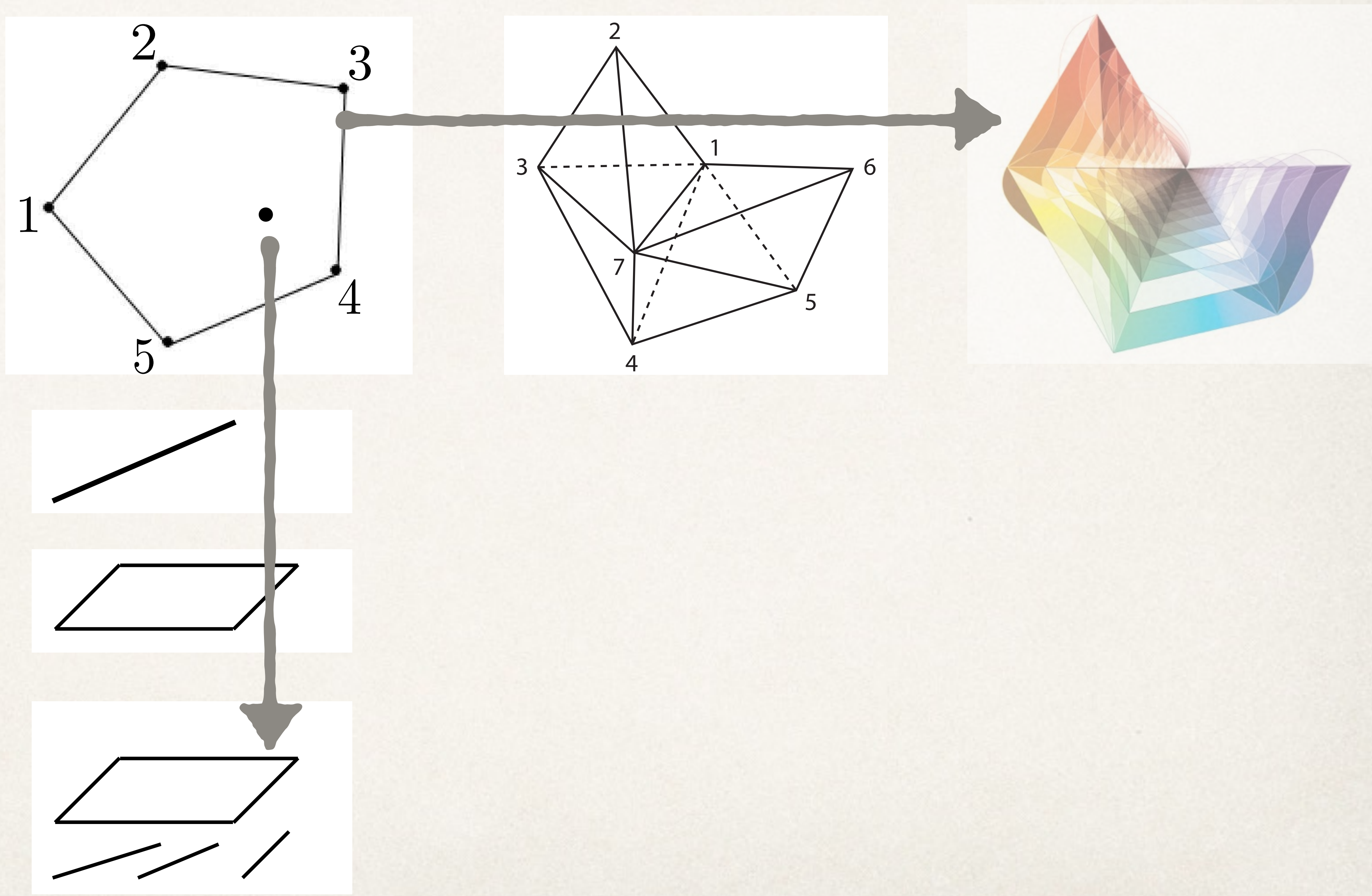
<u>Start:</u>
Point inside a convex polygon





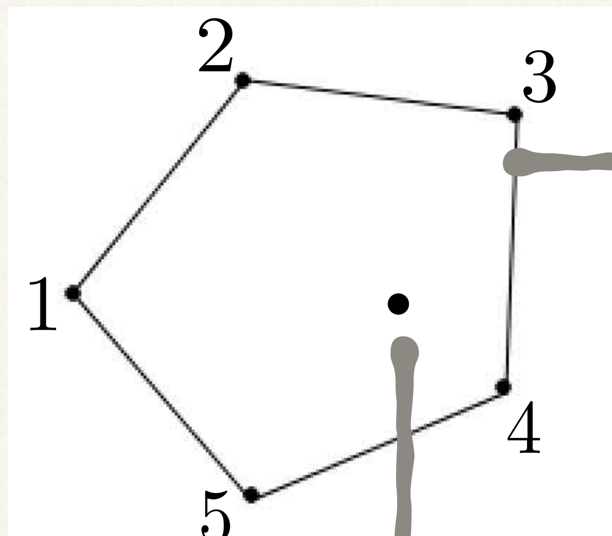
Road to Amplituhedron

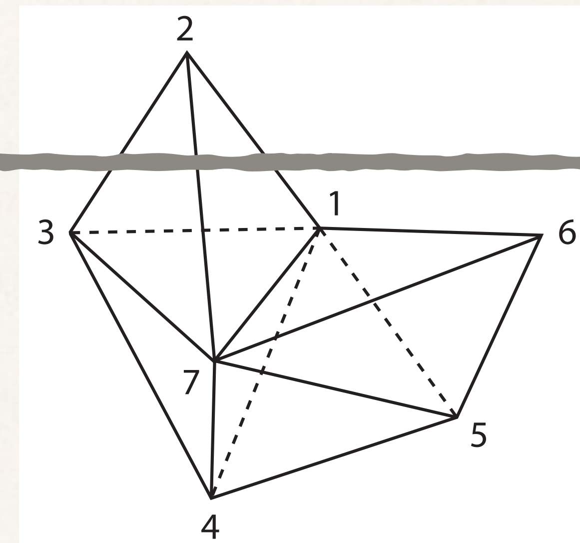
<u>Start:</u>
Point inside a convex polygon

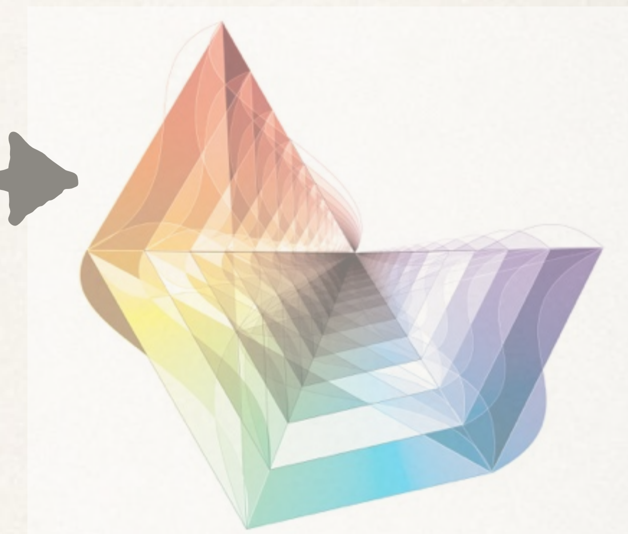


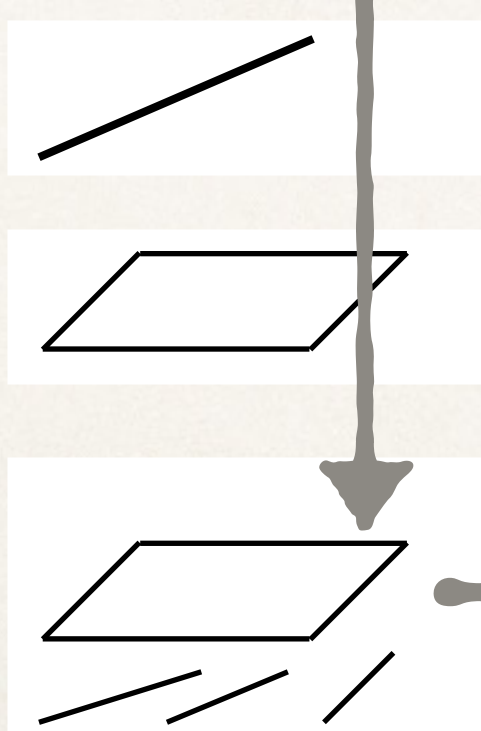
Road to Amplituhedron

<u>Start:</u>
Point inside a convex polygon









<u>Amplituhedron $\mathcal{A}_{n,k,\ell}$</u>

A k-dim plane and ℓ lines inside a $(k+4)$-dim convex space defined by n vertices

Amplituhedron conjecture

- Volume of $\mathcal{A}_{n,k,\ell}$:

Loop integrand in maximally supersymmetric Yang-Mills theory

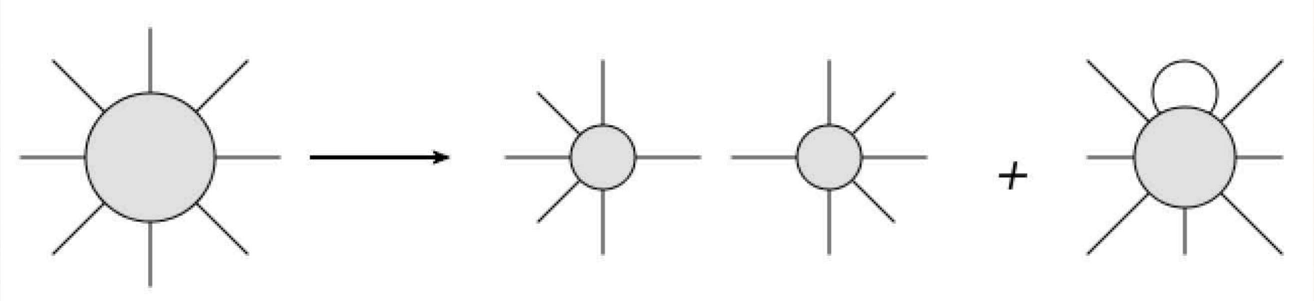
$\ell = 0$: Amplitudes of gluons in QCD

n
number of particles

k
helicity information

ℓ
number of loops

- Consistency check: Locality and Unitarity



- Explicit checks against reference theoretical data

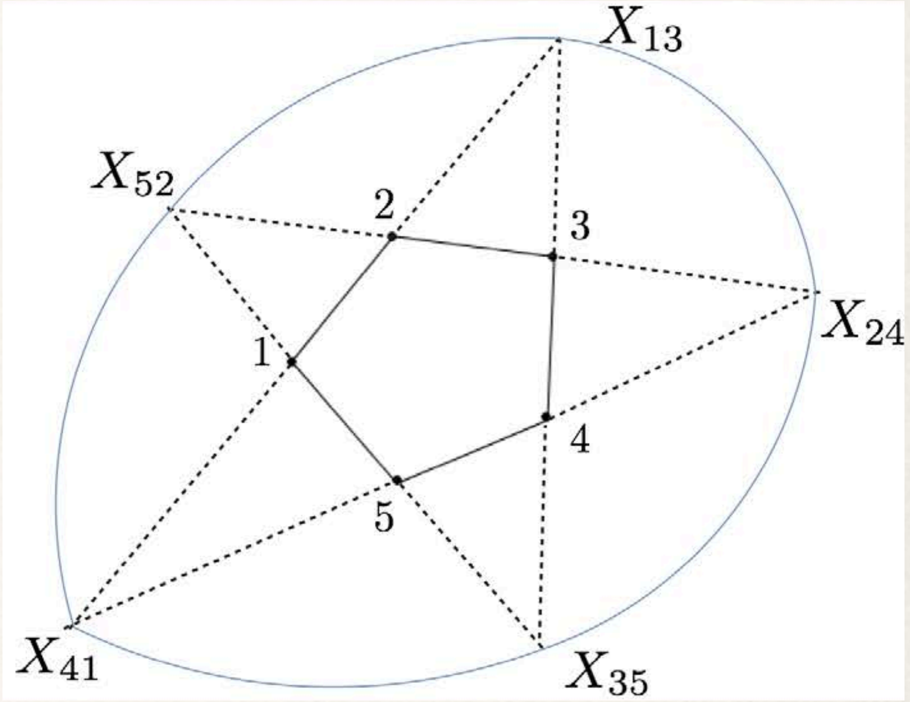
Positivity of amplitudes

(Arkani-Hamed, Hodges, JT, 2014)

- All terms combined in the amplitude

$$I = \frac{(\text{Numerator})}{(\text{all poles})}$$

- Illegal singularities in denominator
- Numerator fixed by **zeroes**
  - Points outside Amplituhedron
  - Canceling higher poles
- Amplitude **positive** (for points inside): volume interpretation

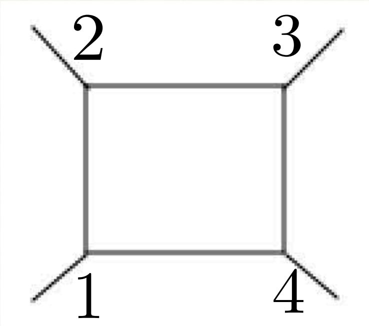


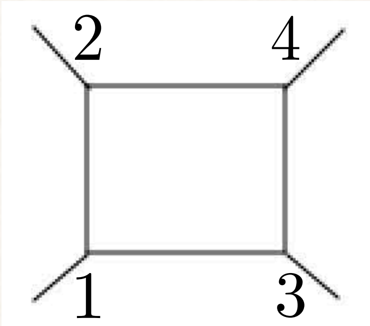
It represents cod-1 surface outside Amplituhedron

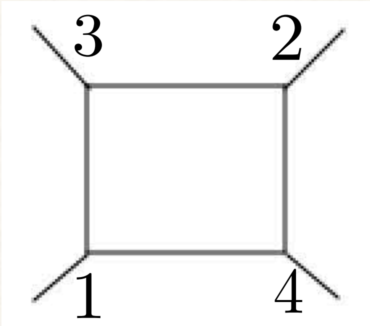
Singularities of non-planar amplitudes

Non-planar amplitudes in N=4 SYM

- No unique integrand, labeling problem



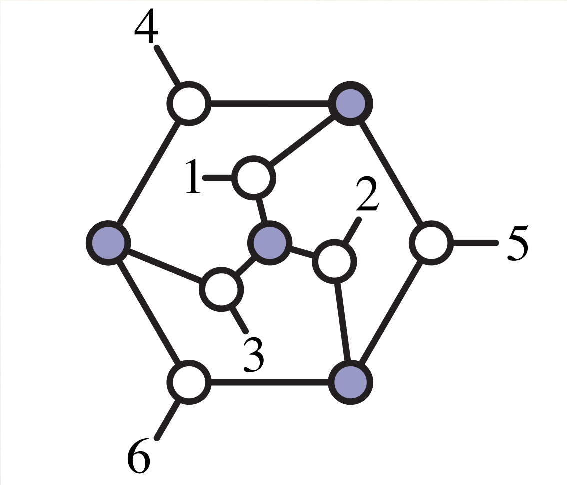




What is ℓ ?

- No momentum twistors, no known symmetries

- On-shell diagrams for singularities



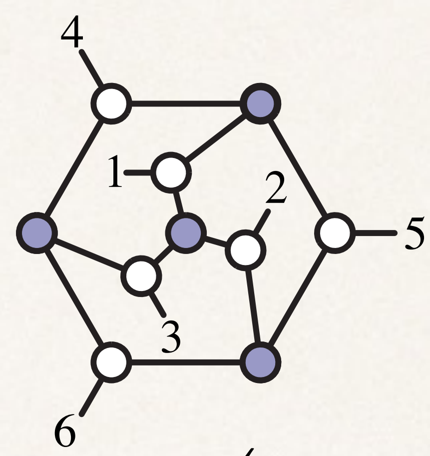
No recursion relations

Non-planar on-shell diagrams

- Non-planar diagrams

(Arkani-Hamed, Bourjaily, Cachazo, Postnikov, JT, 2014)

(Franco, Galloni, Penante, Wen 2015)



Same logarithmic form

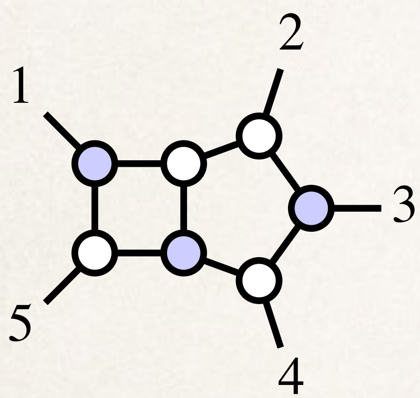
$$C = \begin{pmatrix} * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \end{pmatrix} \in G(3,6)$$

$$\Omega = \frac{d\alpha_1}{\alpha_1}\frac{d\alpha_2}{\alpha_2}\frac{d\alpha_3}{\alpha_3}\frac{d\alpha_4}{\alpha_4}\frac{d\alpha_5}{\alpha_5}\frac{d\alpha_6}{\alpha_6}\delta(C\cdot Z)$$

- **Conjecture**: logarithmic singularities of the amplitude

MHV on-shell diagrams

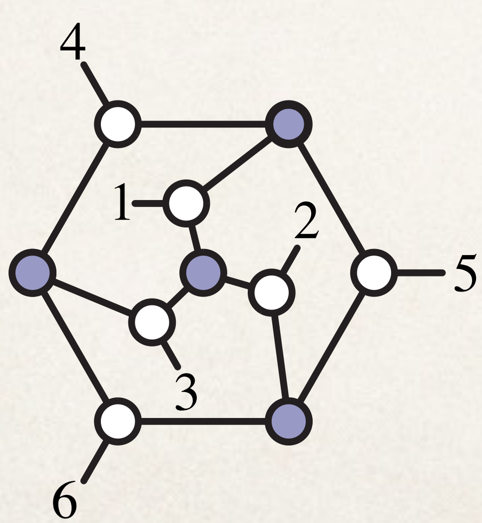
- Planar sector: all are Parke-Taylor factors



$$= \frac{1}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 51\rangle} = PT(12345)$$

required by superconformal symmetry

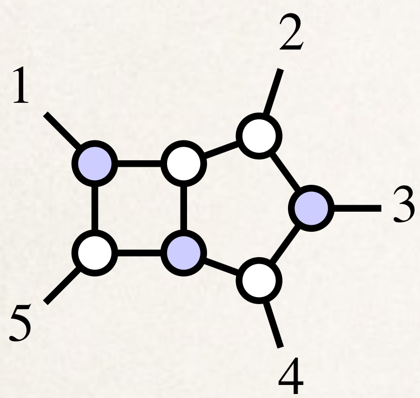
- Non-planar diagrams: holomorphic functions



$$= \frac{(\langle 34\rangle\langle 51\rangle\langle 62\rangle + \langle 14\rangle\langle 25\rangle\langle 63\rangle)^2}{\langle 12\rangle\langle 23\rangle\langle 31\rangle\langle 25\rangle\langle 56\rangle\langle 62\rangle\langle 34\rangle\langle 46\rangle\langle 63\rangle\langle 45\rangle\langle 51\rangle\langle 14\rangle}$$

MHV on-shell diagrams

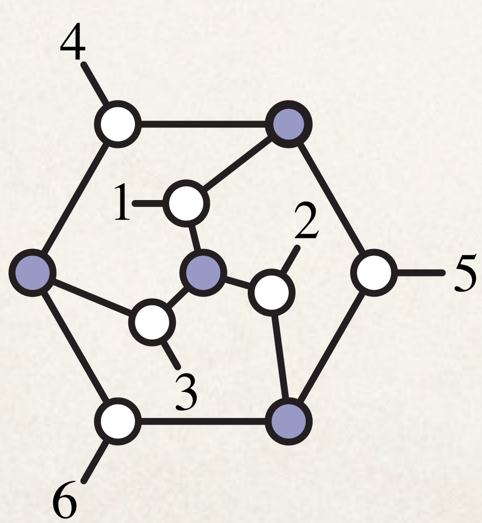
- Planar sector: all are Parke-Taylor factors



$$= \frac{1}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 51\rangle} = PT(12345)$$

required by superconformal symmetry

- Non-planar diagrams: holomorphic functions



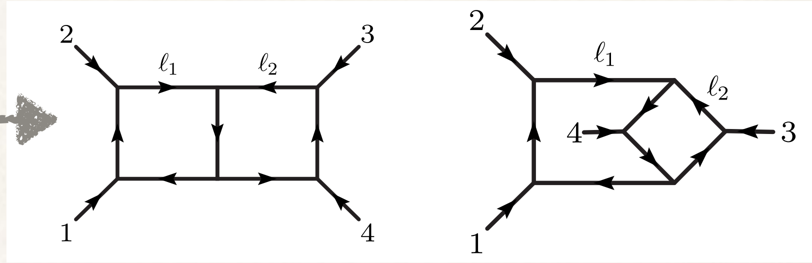
$$= PT(123456) + PT(124563) + PT(142563) + PT(145623)$$
$$+ PT(146235) + PT(146253) + PT(162345)$$

Linear combination of Parke-Taylor factors

Dual conformal symmetry

- Conservative approach: amplitude as a sum of integrals

$$A = \sum_i a_i \cdot C_i \cdot I_i$$



$$f^{1ab} f^{bcd} \dots f^{4ef}$$

- Planar limit: integrals I_i dual conformal invariant (DCI)

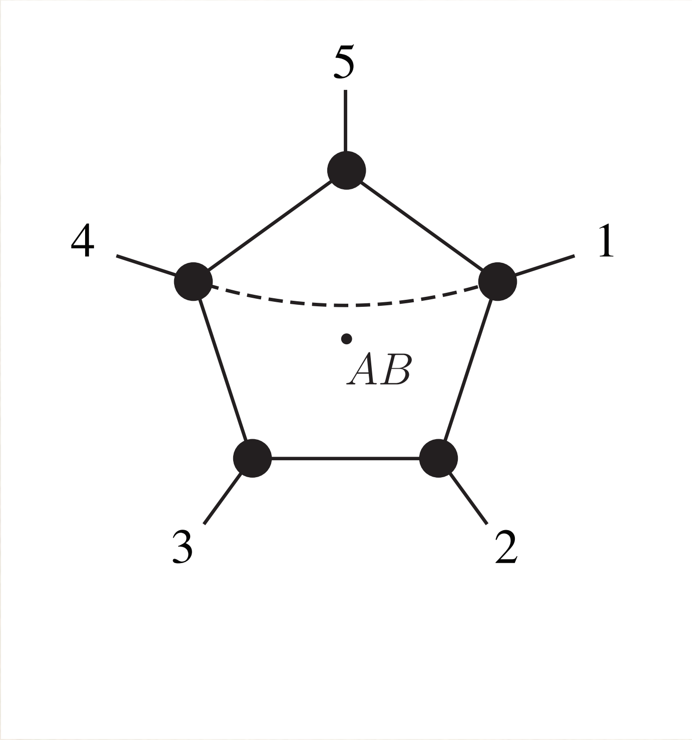
- How to distinguish: DCI vs non-DCI ? → depends on I

  Can we see it in the structure of singularities?

DCI in action

Unit leading singularities

For n>6 can be cross ratio



Chiral vs scalar pentagon

$$\frac{\langle ABd^2A\rangle\langle ABd^2B\rangle\langle AB13\rangle\langle 2345\rangle\langle 4512\rangle}{\langle AB12\rangle\langle AB23\rangle\langle AB34\rangle\langle AB45\rangle\langle AB51\rangle}$$

Unit

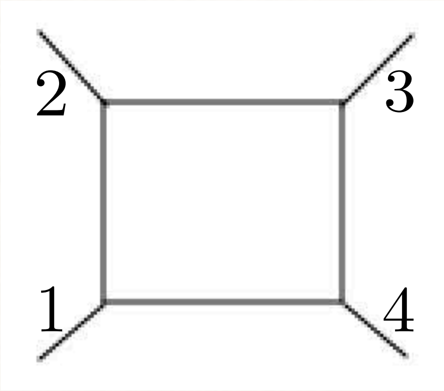
$$\frac{\langle ABd^2A\rangle\langle ABd^2B\rangle\langle ABI\rangle}{\langle AB12\rangle\langle AB23\rangle\langle AB34\rangle\langle AB45\rangle\langle AB51\rangle}$$

Non-unit

On all 4L-cut the residue is 1

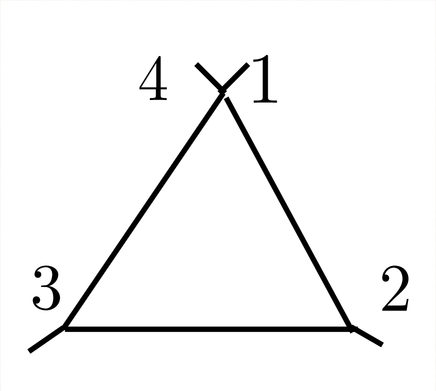
DCI in action

No poles at infinity $\ell \to \infty$



$$\frac{\langle ABd^2A\rangle\langle ABd^2B\rangle\langle 1234\rangle^2}{\langle AB12\rangle\langle AB23\rangle\langle AB34\rangle\langle AB41\rangle}$$

No pole



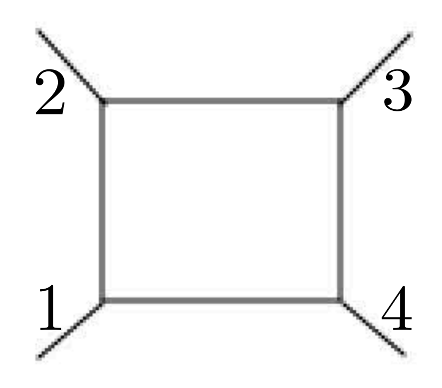
$$\frac{\langle ABd^2A\rangle\langle ABd^2B\rangle\langle 1234\rangle\langle 23I\rangle}{\langle AB12\rangle\langle AB23\rangle\langle AB34\rangle\langle ABI\rangle}$$

Pole

Cut this propagator

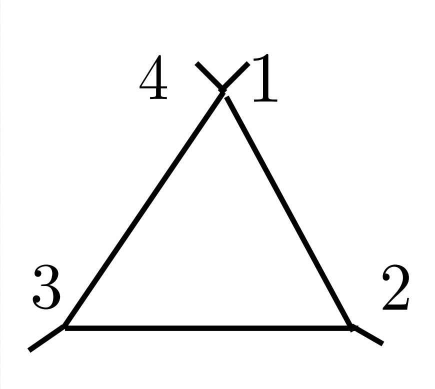
DCI in action

No poles at infinity $\ell \to \infty$



$$\frac{\langle ABd^2A\rangle\langle ABd^2B\rangle\langle 1234\rangle^2}{\langle AB12\rangle\langle AB23\rangle\langle AB34\rangle\langle AB41\rangle}$$

No pole

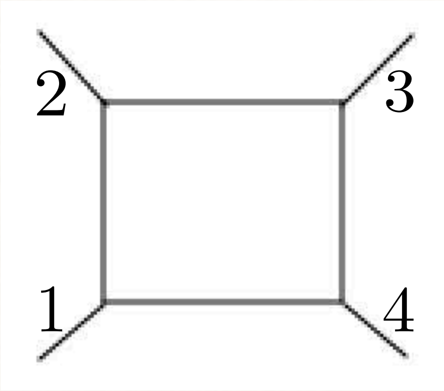


$$\frac{d^4\ell}{\ell^2(\ell+k_2)^2(\ell+k_2+k_3)^2}$$

Pole

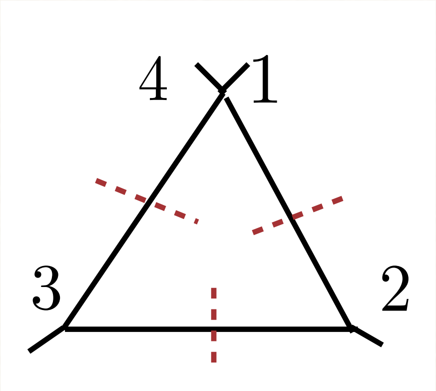
DCI in action

No poles at infinity $\ell \to \infty$



$$\frac{\langle ABd^2A\rangle\langle ABd^2B\rangle\langle 1234\rangle^2}{\langle AB12\rangle\langle AB23\rangle\langle AB34\rangle\langle AB41\rangle}$$

No pole

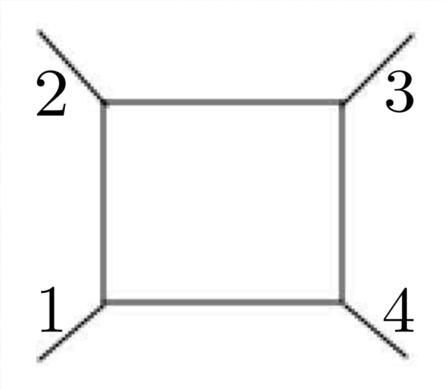


$$\frac{d^4\ell}{\underset{\downarrow\atop 0}{\ell^2}\underset{\downarrow\atop 0}{(\ell+k_2)^2}\underset{\downarrow\atop 0}{(\ell+k_2+k_3)^2}}$$

Pole

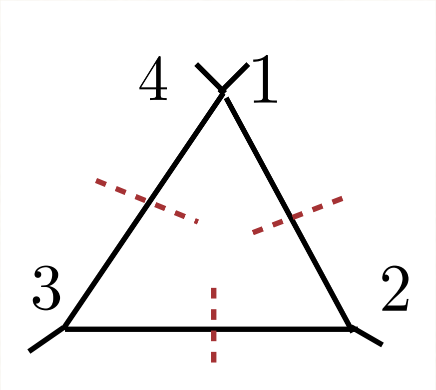
DCI in action

No poles at infinity $\ell \to \infty$



$$\frac{\langle ABd^2A\rangle\langle ABd^2B\rangle\langle 1234\rangle^2}{\langle AB12\rangle\langle AB23\rangle\langle AB34\rangle\langle AB41\rangle}$$

No pole



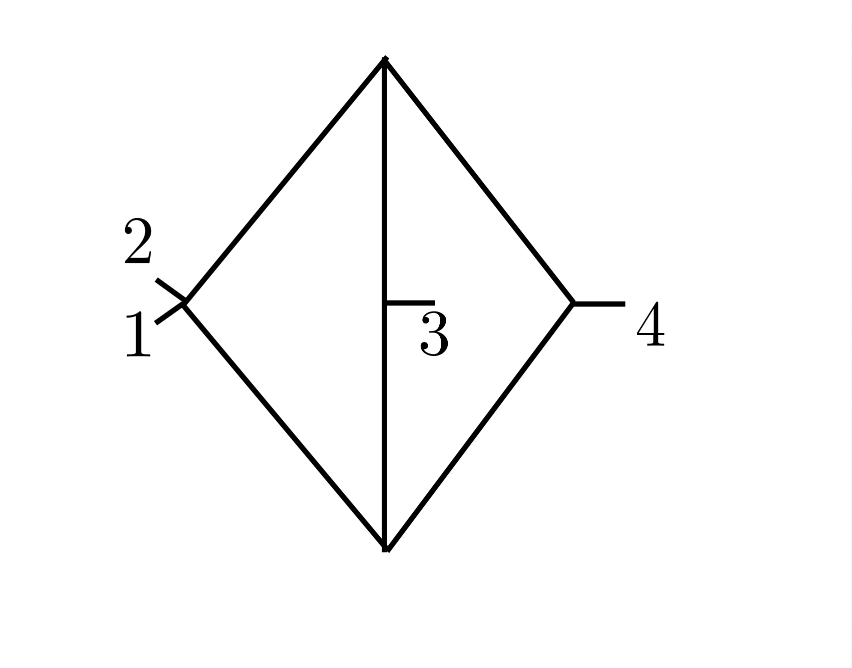
Pole

$$\frac{d\alpha}{\alpha} \qquad \ell + k_2 = \alpha\lambda_2\tilde{\lambda}_3$$

$\alpha \to \infty$ $\ell \to \infty$

Source of poles at infinity

- Planar: triangle sub-diagrams present
- Non-planar: more types of poles at infinity



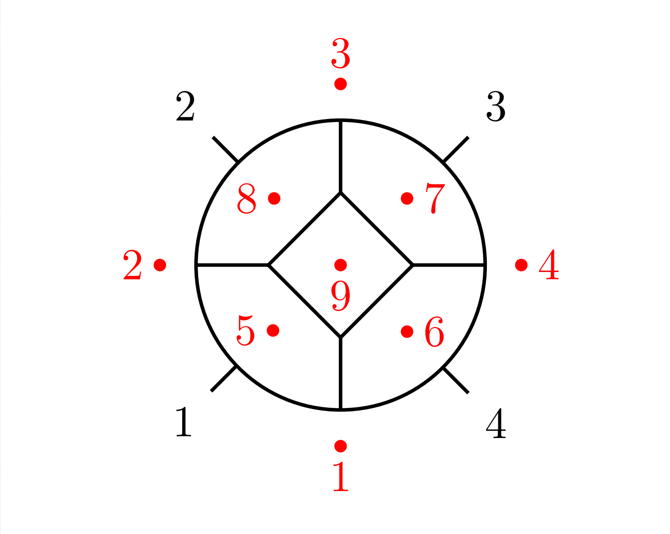
Only box subdiagrams
Has poles at infinity

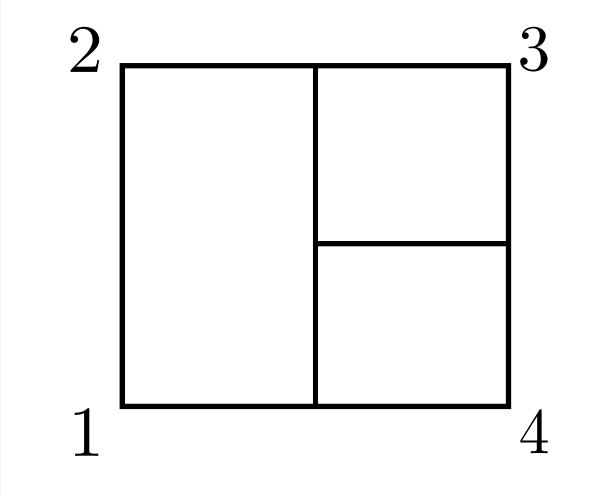
Logarithmic singularities vs DCI

- Multiple poles in the cut structure

$$\text{Integral} \;\rightarrow\; \text{Cut} \rightarrow \;\text{Cut}\; \rightarrow \;\frac{d\alpha}{\alpha^2}$$

- At low loops saved by DCI





$N \sim \langle AB34 \rangle$

Not at higher loops: new condition

Expansion of the amplitude

(Arkani-Hamed, Bourjaily, Cachazo, JT 2014) (Bern, Herrmann, Litsey, Stankowicz, JT 2014, 2015)

- Expansion for the (MHV) amplitude

$$A = \sum_i a_i \cdot C_i \cdot I_i$$

Parke-Taylor factors (a_i) — Color factors (C_i) — Basis of integrals (I_i)

- **Conjecture:** term-by-term
  - Logarithmic singularities
  - No poles at infinity
  - Unit leading singularities

→ After integration:

  - Uniform transcendentality
  - No prefactors
  - Special cross-ratios…

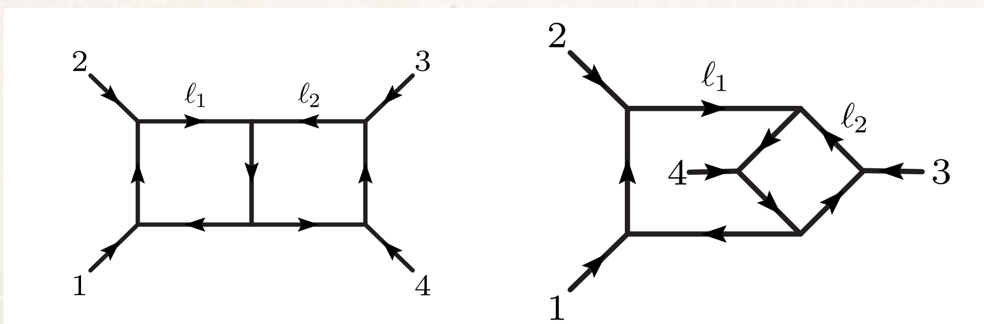
- Uniform transcendental non-planar integrals studied

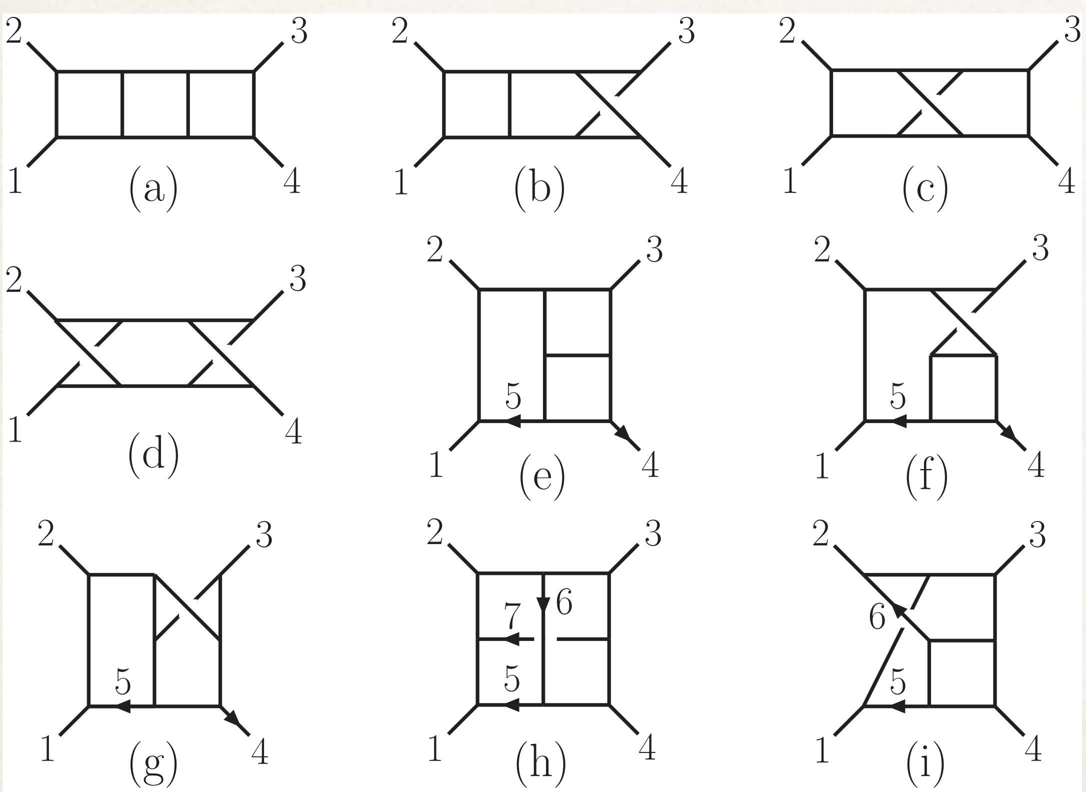
(Gehrmann, Henn, Huber 2011, Henn 2014, Henn, Smirnov, Smirnov 2015)

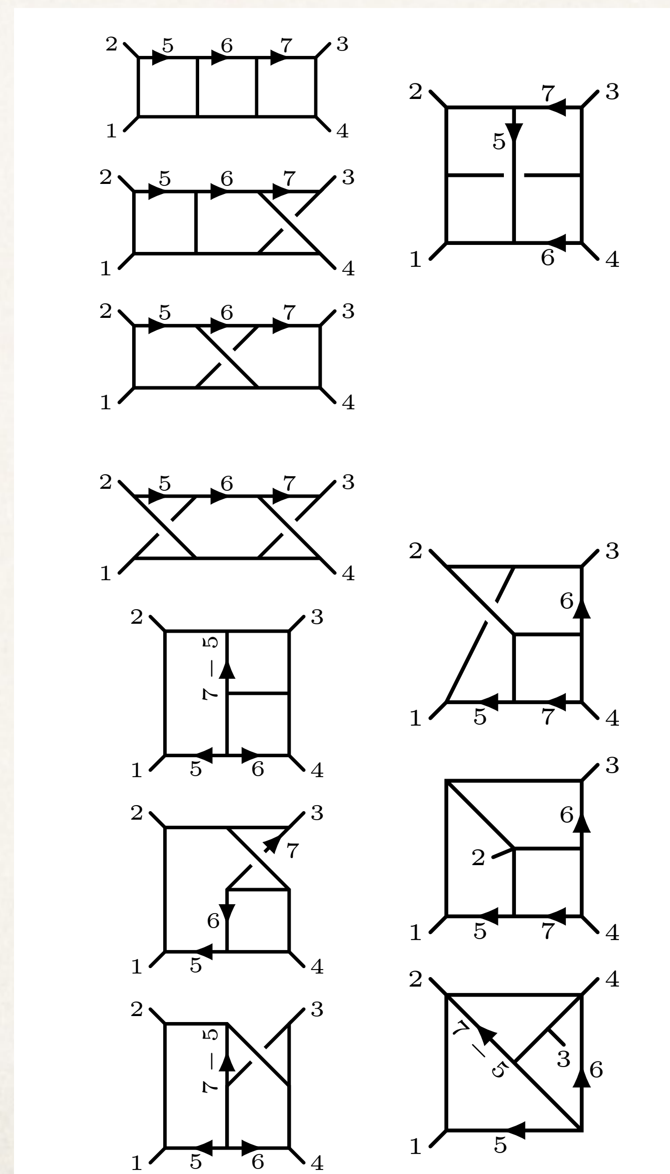
Explicit checks

(Arkani-Hamed, Bourjaily, Cachazo, JT, 2014) (Bern, Herrmann, Litsey, Stankowicz, JT 2014, 2015)

- Construct basis for 4pt at 2-loop and 3-loop, 5pt 2-loop





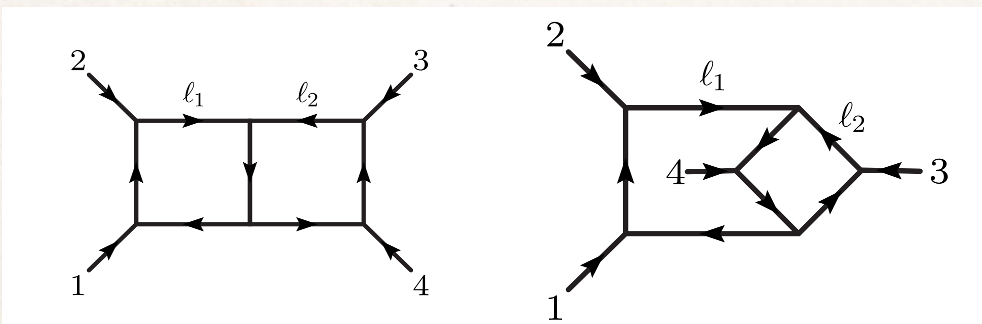


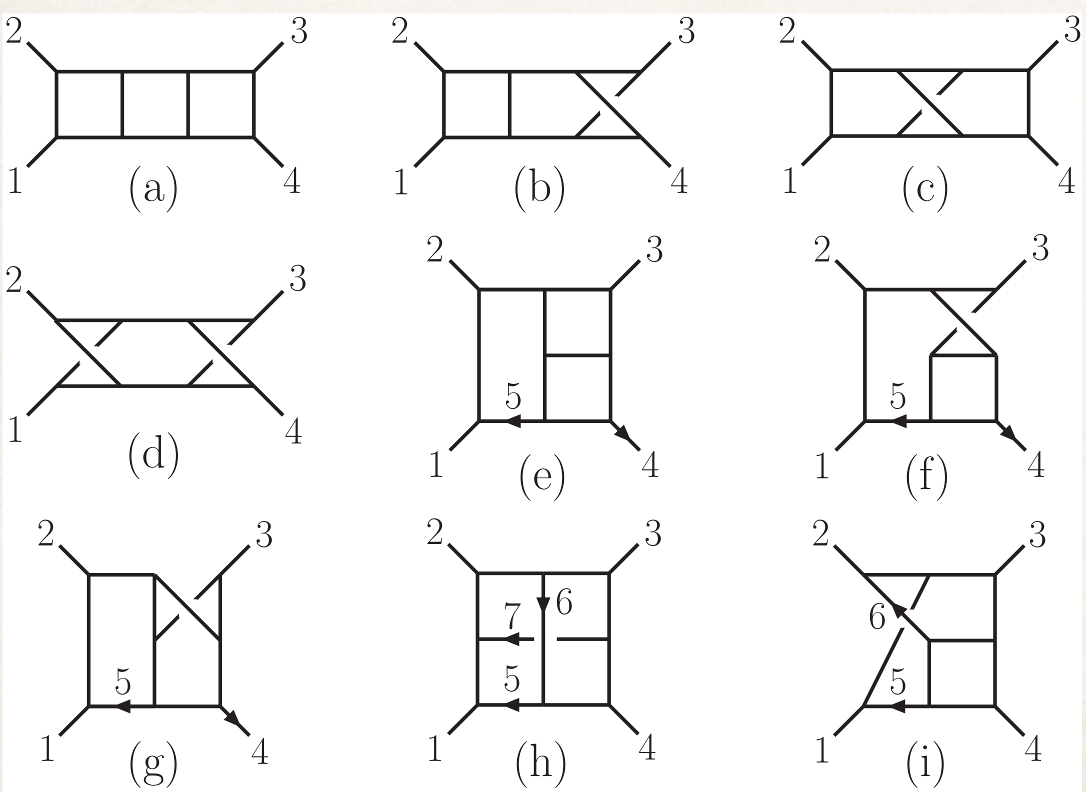
Expand the amplitude:

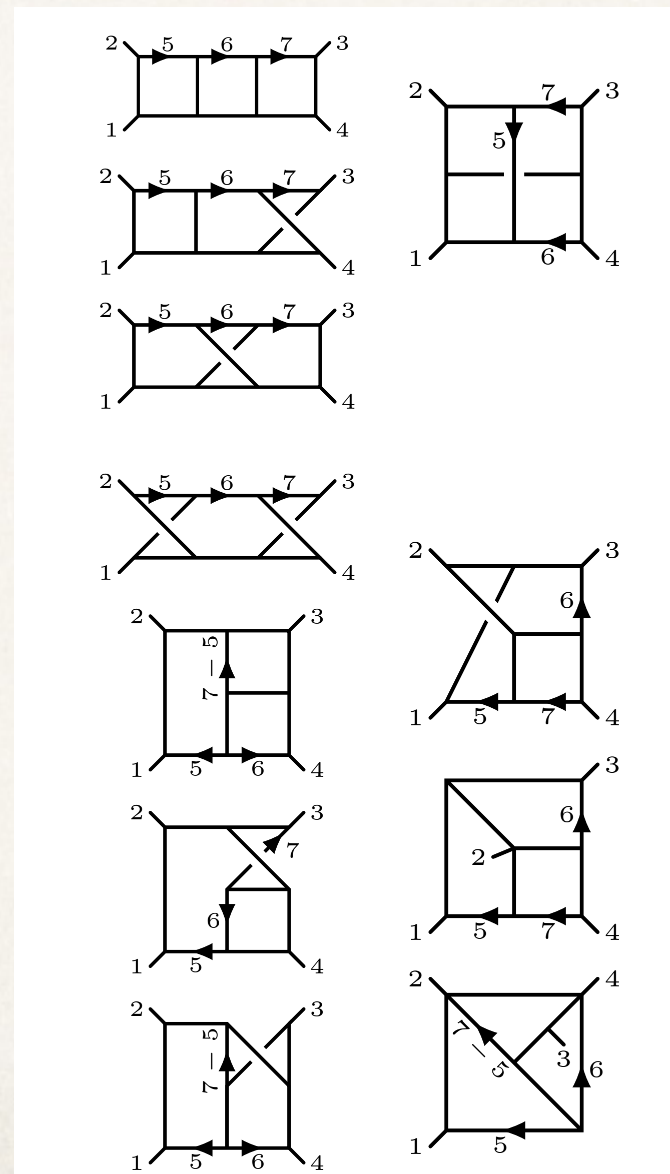
Explicit checks

(Arkani-Hamed, Bourjaily, Cachazo, JT, 2014) (Bern, Herrmann, Litsey, Stankowicz, JT 2014, 2015)

- Construct basis for 4pt at 2-loop and 3-loop, 5pt 2-loop







Expand the amplitude:

Coefficient fixed by standard unitarity methods

Coefficients from zeroes

(Bern, Herrmann, Litsey, Stankowicz, JT 2015)

- Go even further in the analogy with planar
- Use only equations of type $\boxed{\text{Cut } I = 0}$
  - Illegal cuts: non-MHV or spurious cuts
  - No target (product of trees) necessary
- In planar: conjecture, evidence for dual Amplituhedron
- Conjecture: $$A = \sum_i a_i \cdot C_i \cdot I_i$$ Test for our three cases:

Fixed by vanishing cuts

Coefficients from zeroes

(Bern, Herrmann, Litsey, Stankowicz, JT 2015)

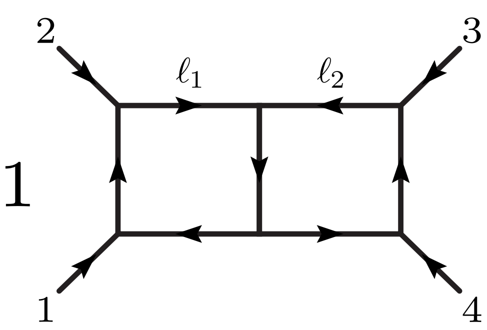
- Go even further in the analogy with planar
- Use only equations of type $\boxed{\text{Cut } I = 0}$
  - Illegal cuts: non-MHV or spurious cuts
  - No target (product of trees) necessary
- In planar: conjecture, evidence for dual Amplituhedron
- Conjecture: $$A = \sum_i a_i \cdot C_i \cdot I_i$$

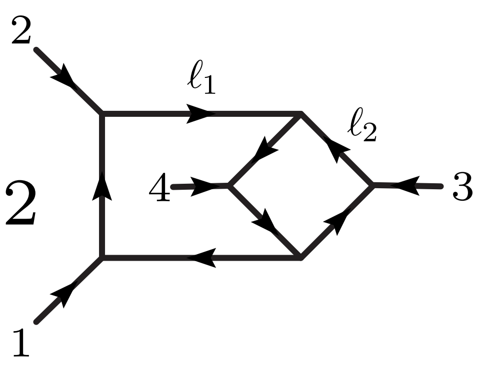
Fixed by vanishing cuts (a_i)

Test for our three cases: ✓

Example of zero condition

- Expansion of the amplitude

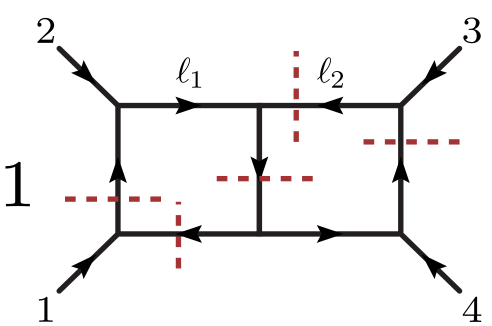
$$M_2 = \sum_{\sigma} \quad a_1$$ 

$$+ \quad a_2$$ 

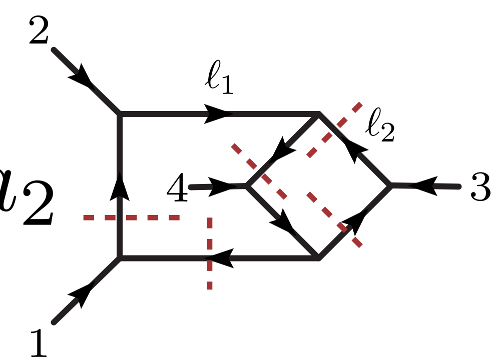
Example of zero condition

- Expansion of the amplitude

$$\text{Cut } M_2 = \sum_{\sigma} \quad a_1$$

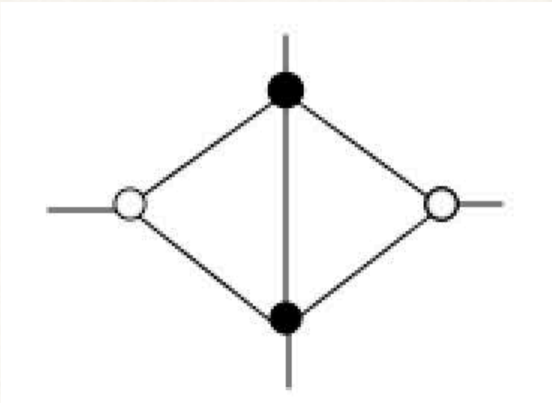


$$+ \quad a_2$$



$$= 0$$

Illegal 5-cut



$k = 1$

Fixes relative coefficient

$$a_1 = a_2$$

Non-planar N=4 conclusion

- Amplitudes (integrands) in complete N=4 SYM:

Analogue of dual conformal symmetry
On-shell diagrams / Amplituhedron insights

- No poles at infinity
- Special leading singularities
- Logarithmic singularities
- Zero conditions

- Homogeneous conditions define the amplitude

This is begging for **geometric/volume interpreation**
Role of color factors?

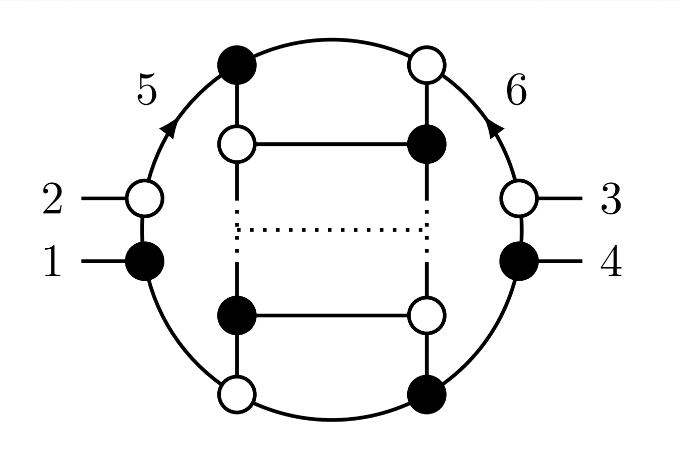
Few words about supergravity amplitudes

Similarities with Yang-Mills

- BCJ relations

$$A^{(YM)} = \sum_j \frac{n_j c_j}{s_j} \rightarrow A^{(GR)} = \sum_j \frac{n_j^2}{s_j}$$

- Squaring has dramatic consequences on singularities
- Loop amplitudes in N=8 SUGRA: poles at infinity

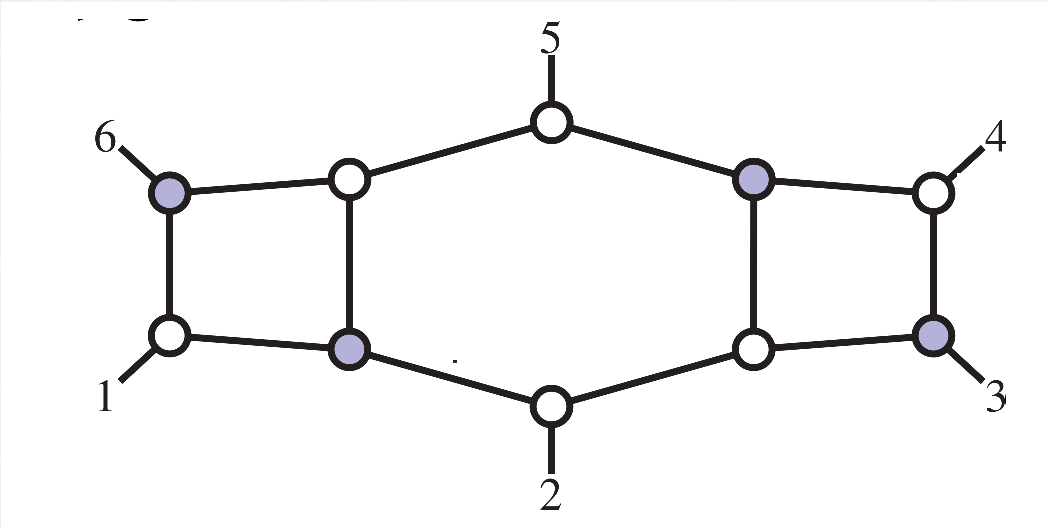


Bad UV behavior
of integrals
in the amplitude

Gravity on-shell diagrams

(Herrmann, JT, in progress)

- Well defined on-shell objects
- No recursion relations, capture singularity structure
- MHV on-shell diagrams: not holomorphic in N=8



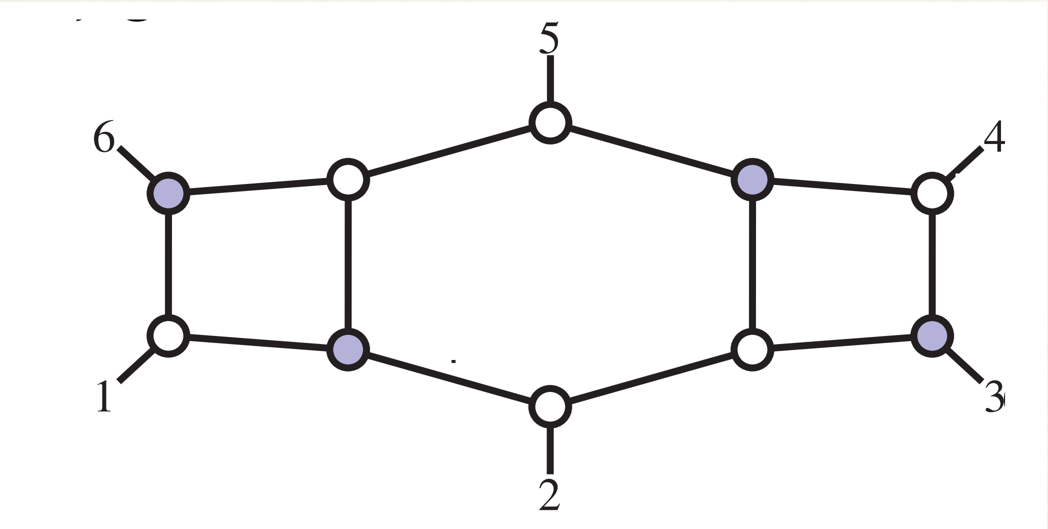
$$= \frac{1}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 45\rangle\langle 56\rangle\langle 61\rangle}$$

in N=4 SYM

Gravity on-shell diagrams

(Herrmann, JT, in progress)

- Well defined on-shell objects
- No recursion relations, capture singularity structure
- MHV on-shell diagrams: not holomorphic in N=8



$$= \frac{\langle 5|1+6|2]\langle 2|3+4|5][16]^2[34]^2}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 45\rangle\langle 56\rangle\langle 61\rangle\langle 25\rangle^2}$$

in N=8 SUGRA

Gravity on-shell diagrams

(Herrmann, JT, in progress)

- Default way to calculate: product of 3pt amplitudes
- Amalgamation of 3pt vertices: Grassmannian formula

  Does not preserve logarithmic or any other nice form
- Rules to define the form globally needed

$$\Omega = F(\alpha)\, \delta(C \cdot Z)$$

↓

We start to understand how it looks like → Dramatic implications for singularities of gravity amplitudes

Conclusion

Conclusion

- Planar N=4 SYM: on-shell diagrams, Amplituhedron
- Non-planar N=4 SYM: same properties seem to hold

  Evidence for non-planar geometric construction → Good variables needed

- Gravity in progress

Thank you for your attention