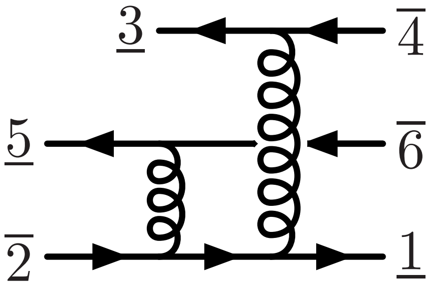
Hidden Simplicity in QCD and Gravity Amplitudes

Henrik Johansson

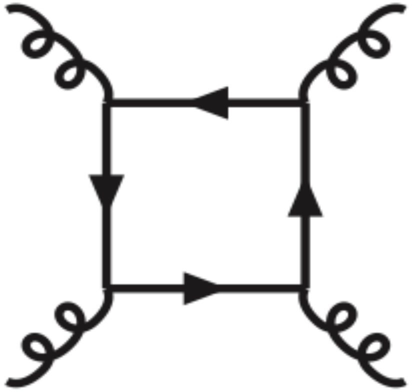
Uppsala U. & Nordita



March 17, 2016

MHV @ 30

Fermilab



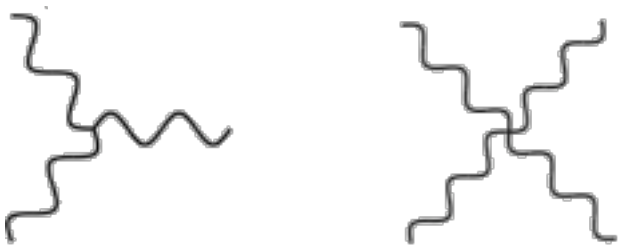
work with Alexander Ochirov [1407.4772, 1507.00332]
and Marco Chiodaroli, Murat Gunaydin, Radu Roiban
[1408.0764, 1511.01740, 1512.09130]

Gauge and gravity theories

A story of Hidden Simplicity using basic QFT properties:

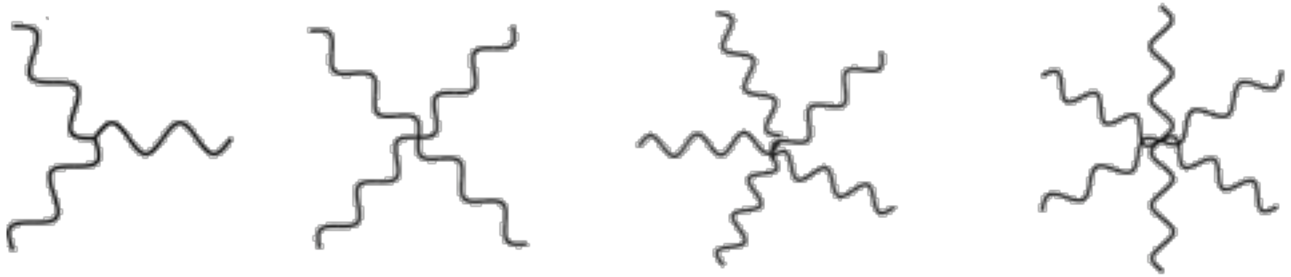
Gauge theories: massless spin-1, gauge invariance, color, …

A_μ



Gravity theories: massless spin-2, diffeomorphism invariance, …

$h_{\mu\nu}$



…

→ observe striking simplicity in general structure of these theories

Amplitudes in a gauge theory

kinematic numerators

color factors

propagators

cubic diagram form:

$$\mathcal{A}^{\rm tree} = \sum_{i\in{\rm cubic}} \frac{n_i c_i}{D_i}$$

$n_i \equiv \varepsilon_\mu(p)\, n_i^\mu$ Consider a gauge transformation $\varepsilon \to \varepsilon + \alpha p$

$$n_i \to n_i + \Delta_i \qquad \Delta_i = \alpha\, p_\mu n_i^\mu$$

Invariance of $\mathcal{A}^{\rm tree}$ requires that c_i are linearly dependent

$c_i - c_j = c_k$ [Jacobi id. or Lie alg. commutation]

we automatically have:

$$\sum_{i\in{\rm cubic}} \frac{\Delta_i c_i}{D_i} = 0$$

Build gravity amplitudes

Assume the gauge freedom can be exploited to find numerators

$$c_i - c_j = c_k \quad \Leftrightarrow \quad n_i - n_j = n_k$$

dual to the color factors

Then the double copy $\mathcal{M}^{\text{tree}} = \sum_{i \in \text{cubic}} \frac{n_i \tilde{n}_i}{D_i}$ → Gravity

describes a spin-2 theory $\varepsilon_{\mu\nu} = \varepsilon_\mu \varepsilon_\nu$

invariant under (linear) diffeos $\varepsilon_{\mu\nu} \to \varepsilon_{\mu\nu} + p_\mu \xi_\nu + \xi_\mu p_\nu$

$$\mathcal{M}^{\text{tree}} \to \mathcal{M}^{\text{tree}} + \underbrace{\sum_{i \in \text{cubic}} \frac{\Delta_i \tilde{n}_i}{D_i} + \sum_{i \in \text{cubic}} \frac{n_i \tilde{\Delta}_i}{D_i}}_{= 0}$$

Outline

- Motivation & review: color-kinematics duality
  - Various gravity/gauge theories
- Generalization to QCD tree amplitudes
- New color decomposition
- Primitive amplitude relations for QCD
- Double copies of QCD
- Simple one loop application: 1-loop 4pt
- Conclusion

Color-kinematics duality

Color-kinematics duality for pure (S)YM

YM theories are controlled by a hidden kinematic Lie algebra

- Amplitude expanded in terms of cubic graphs:

$$\mathcal{A}_n^{(L)} = \sum_{i\in\Gamma_3}\int \frac{d^{LD}\ell}{(2\pi)^{LD}}\frac{1}{S_i}\frac{n_i c_i}{p_{i_1}^2 p_{i_2}^2 p_{i_3}^2\cdots p_{i_l}^2}$$

kinematic numerators

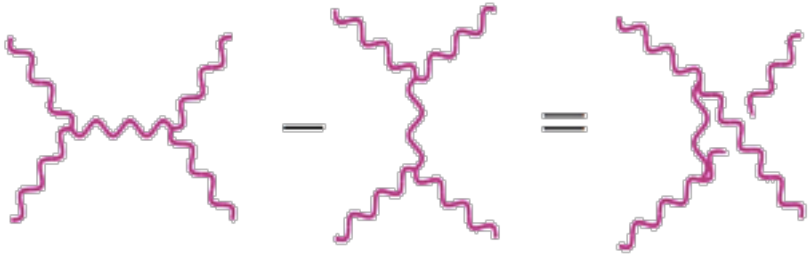
color factors

propagators

Color & kinematic numerators satisfy same relations:

Bern, Carrasco, HJ

$$n_i - n_j = n_k \quad \Leftrightarrow \quad c_i - c_j = c_k$$

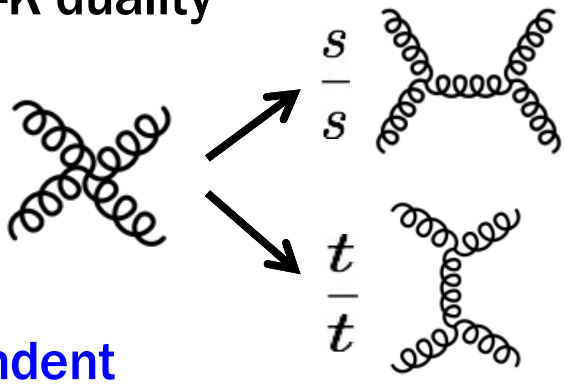


Jacobi identity

$$f^{dac}f^{cbe} - f^{dbc}f^{cae} = f^{abc}f^{dce} \quad \Rightarrow \quad c_i - c_j = c_k$$

Generalized gauge transformations

In general Feynman diagrams do not obey C-K duality



- Four-gluon vertex absorbed into cubic graphs → ambiguity

- Feynman diagrams are gauge-dependent
  → no reason to expect C-K duality to be present in all gauges

Amplitudes are invariant under "generalized gauge transformations"

$$n_i \to n_i + \Delta_i \qquad \text{such that} \qquad \sum_i \frac{c_i \Delta_i}{\prod_\alpha p_\alpha^2} = 0$$

but not duality: $n_i - n_j \overset{?}{=} n_k \quad \Leftrightarrow \quad c_i - c_j = c_k$

Claim: starting from a general gauge there exists transformations Δ_i *that makes the numerators obey the duality !*

Bern, Carrasco, HJ ('08 - '10)　　shown → Lee, Mafra, Schlotterer ('15)

Gauge-invariant relations (pure glue)

$$A(1,2,\ldots,n-1,n) = A(n,1,2,\ldots,n-1)$$ cyclicity → (*n*-1)! basis

$$\sum_{i=1}^{n-1} A(1,2,\ldots,i,n,i+1,\ldots,n-1) = 0$$

U(1) decoupling
Mangano, Parke, Xu

$$A(1,\beta,2,\alpha) = (-1)^{|\beta|} \sum_{\sigma\in\alpha\sqcup\!\sqcup\beta^T} A(1,2,\sigma)$$

Kleiss-Kuijf
relations (‘89)

(*n*-2)! basis

$$\sum_{i=2}^{n-1}\Big(\sum_{j=2}^{i} s_{jn}\Big) A(1,2,\ldots i,n,i+1,\ldots,n-1) = 0$$

$$A(1,2,\alpha,3,\beta) = \sum_{\sigma\in S(\alpha)\sqcup\!\sqcup\beta} A(1,2,3,\sigma) \prod_{i=1}^{|\alpha|} \frac{\mathcal{F}(3,\sigma,1|i)}{s_{2,\alpha_1,\ldots,\alpha_i}}$$

BCJ relations (‘08)
(*n*-3)! basis

BCJ rels. proven via string theory by Bjerrum-Bohr, Damgaard, Vanhove; Stieberger (’09)

and field theory proofs through BCFW: Feng, Huang, Jia; Chen, Du, Feng (’10 -’11)

Relations used in string calcs: Mafra, Stieberger, Schlotterer, et al. (’11 -’15)

Relations used by Cachazo, He, Yuan to motivate CHY and scattering eqns (’13)

Gravity is a double copy of YM

Gravity amplitudes obtained by replacing color with kinematics

$$\mathcal{A}_m^{(L)} = \sum_{i\in\Gamma_3} \int \frac{d^{LD}\ell}{(2\pi)^{LD}} \frac{1}{S_i} \frac{n_i c_i}{p_{i_1}^2 p_{i_2}^2 p_{i_3}^2 \cdots p_{i_l}^2}$$

double copy

Bern, Carrasco, HJ

$$\mathcal{M}_m^{(L)} = \sum_{i\in\Gamma_3} \int \frac{d^{LD}\ell}{(2\pi)^{LD}} \frac{1}{S_i} \frac{n_i \tilde{n}_i}{p_{i_1}^2 p_{i_2}^2 p_{i_3}^2 \cdots p_{i_l}^2}$$

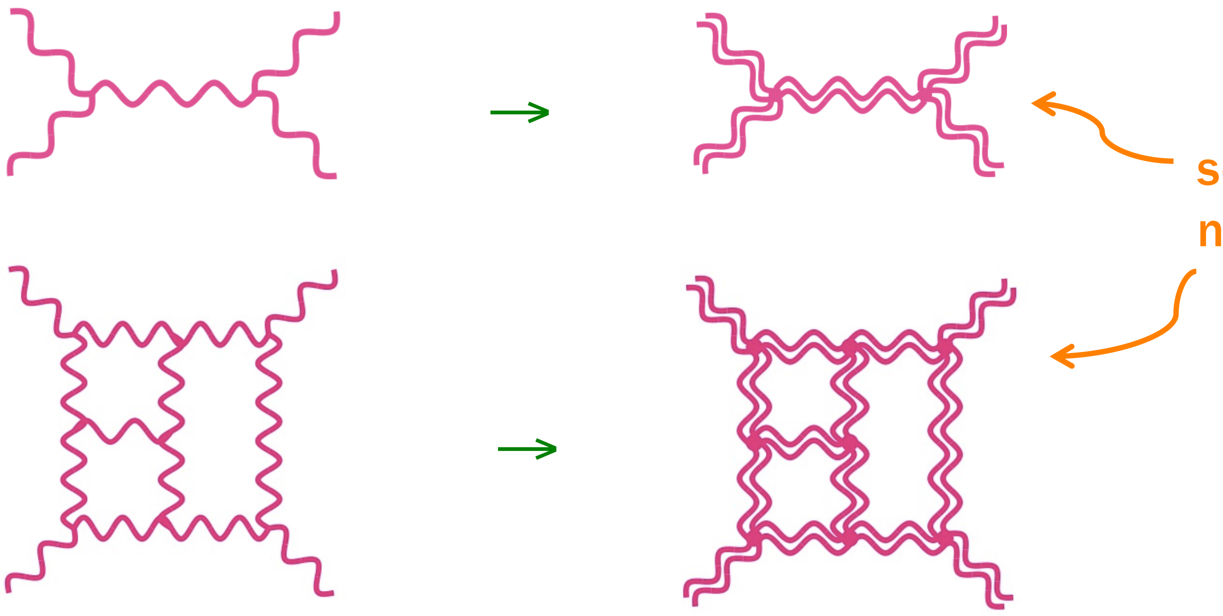
- The two numerators can differ by a generalized gauge transformation
  → only one copy needs to satisfy the kinematic algebra
- The two numerators can differ by the external/internal states
  → graviton, dilaton, axion (*B*-tensor), matter amplitudes
- The two numerators can belong to different theories
  → give a host of different gravitational theories

equivalent to
KLT at tree level
for adj. reps

Squaring of YM theory

Gravity processes = squares of gauge theory ones - entire S-matrix

Bern, Carrasco, HJ ('10)



Yang-Mills

Gravity

squared numerators

E.g. pure Yang-Mills → Einstein gravity + dilaton + axion

$\mathcal{N}$=4 super-YM → $\mathcal{N}$=8 supergravity

Which "gauge" theories obey C-K duality

- Pure $\mathcal{N}$=0,1,2,4 super-Yang-Mills (any dimension) — Bern, Carrasco, HJ ('08) Bjerrum-Bohr, Damgaard, Vanhove; Stieberger; Feng et al. Mafra, Schlotterer, etc ('08-'11)
- Self-dual Yang-Mills theory O'Connell, Monteiro ('11)
- Heterotic string theory Stieberger, Taylor ('14)
- Yang-Mills + F^3 theory Broedel, Dixon ('12)
- QCD, super-QCD, higher-dim QCD HJ, Ochirov ('15)
- Generic matter coupled to $\mathcal{N}$= 0,1,2,4 super-Yang-Mills — Chiodaroli, Gunaydin, Roiban; HJ, Ochirov ('14)
- Spontaneously broken $\mathcal{N}$= 0,2,4 SYM Chiodaroli, Gunaydin, HJ, Roiban ('15)
- Yang-Mills + scalar ϕ^3 theory Chiodaroli, Gunaydin, HJ, Roiban ('14)
- Bi-adjoint scalar ϕ^3 theory — Bern, de Freitas, Wong ('99), Bern, Dennen, Huang; Du, Feng, Fu; Bjerrum-Bohr, Damgaard, Monteiro, O'Connell
- NLSM/Chiral Lagrangian Chen, Du ('13)
- *D*=3 Bagger-Lambert-Gustavsson theory (Chern-Simons-matter) Bargheer, He, McLoughlin; Huang, HJ, Lee ('12 -'13)

Which “gravity” theories are double copies

- Pure $\mathcal{N}$=4,5,6,8 supergravity (2 < D < 11) KLT (‘86), Bern, Carrasco, HJ (’08 -’10)
- Einstein gravity and pure $\mathcal{N}$=1,2,3 supergravity HJ, Ochirov (’14)
- Self-dual gravity O’Connell, Monteiro (’11)
- Closed string theories Mafra, Schlotterer, Stieberger (’11); Stieberger, Taylor (’14)
- Einstein + R^3 theory Broedel, Dixon (’12)
- Abelian matter coupled to supergravity $\left\{\begin{array}{l}\text{Carrasco, Chiodaroli, Gunaydin, Roiban ('12)}\\ \text{HJ, Ochirov ('14 - '15)}\end{array}\right.$
- Magical sugra, homogeneous sugra Chiodaroli, Gunaydin, HJ, Roiban (’15)
- SYM coupled to supergravity Chiodaroli, Gunaydin, HJ, Roiban (’14)
- Spontaneously broken YM-Einstein gravity Chiodaroli, Gunaydin, HJ, Roiban (’15)
- D=3 supergravity (BLG Chern-Simons-matter theory)^2 $\left\{\begin{array}{l}\text{Bargheer, He, McLoughlin;}\\ \text{Huang, HJ, Lee ('12 -'13)}\end{array}\right.$
- Born-Infeld, DBI, Galileon theories (CHY form) Cachazo, He, Yuan (’14)

Color-Kinematics Duality for QCD

Defining QCD

‘QCD’ is taken to be the following theory:

$$\mathrm{SU}(N_c)\ \mathrm{YM}\ +\ N_f\ \text{massive quarks}$$

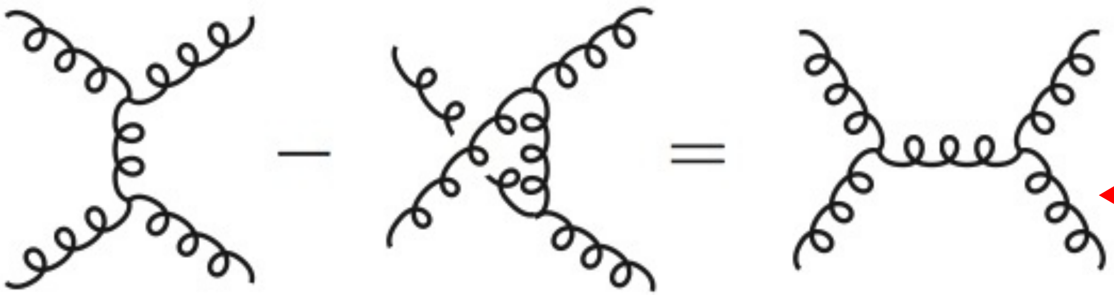
In fact, everything I say will also apply to:

$$G_c\ \mathrm{YM}\ +\ N_f\ \text{massive complex-rep. fermions}\ /\text{scalars}$$

in D dimensions or SUSY extended SQCD

Only use two Lie-algebra properties

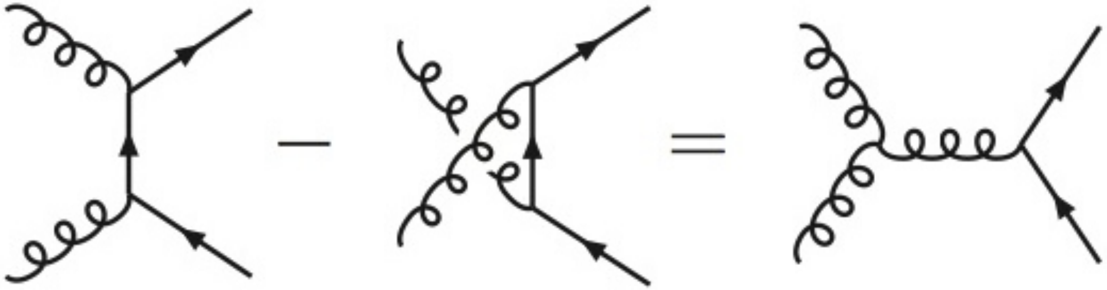
Jacobi Id.



adjoint repr.
or gluon, or
vector multipl.

$$\tilde{f}^{dac}\,\tilde{f}^{cbe} - \tilde{f}^{dbc}\,\tilde{f}^{cae} = \tilde{f}^{abc}\,\tilde{f}^{dce}$$

Commutation Id.



fund. repr.
or fermion, or
complex scalar,
or matter multipl.

$$T^{a}_{i\bar{k}}\,T^{b}_{k\bar{j}} - T^{b}_{i\bar{k}}\,T^{a}_{k\bar{j}} = \tilde{f}^{abc}\,T^{c}_{i\bar{j}}\,.$$

Duality: $n_i - n_j = n_k \quad \Leftrightarrow \quad c_i - c_j = c_k$

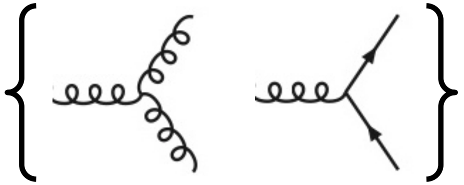
Amplitude presentation for QCD

QCD amplitude with *k* quark lines of distinct flavor:

HJ, Ochirov

$$\mathcal{A}^{(L)}_{n,k} = \sum_i \int \frac{d^{LD}\ell}{(2\pi)^{LD}} \frac{1}{S_i} \frac{n_i c_i}{D_i}$$

sum is over all cubic gluon–quark graphs with vertices

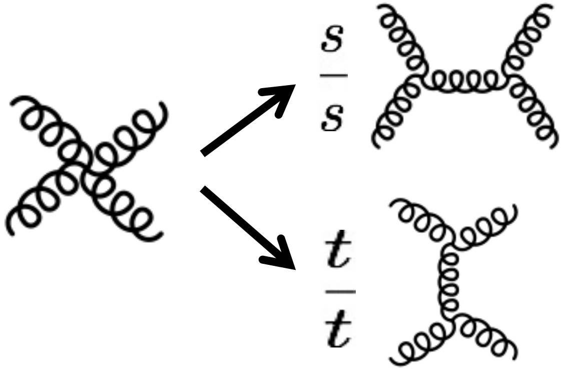


Color factors c_i are built out of f^{abc}, $T^a_{i\bar{\jmath}}$

Number of cubic tree-level graphs

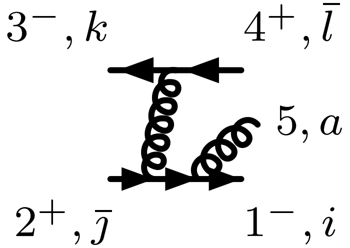
$k \setminus n$	3	4	5	6	7	8
0	1	3	15	105	945	10395
1	1	3	15	105	945	10395
2	-	1	5	35	315	3465
3	-	-	-	7	63	693
4	-	-	-	-	-	99

$$\nu(n,k) = \frac{(2n-5)!!}{(2k-1)!!} \text{ for } 2k \le n$$

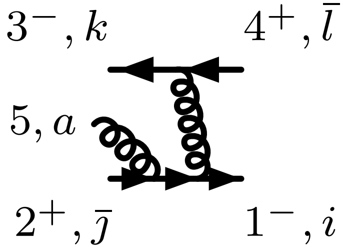


n=5 *k*=2 example

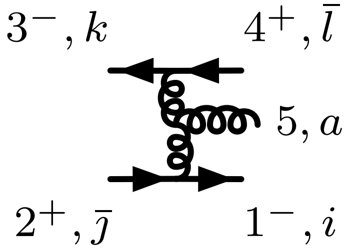
Look at 3 Feynman diagrams out of 5 in total:



$$= \frac{i}{\sqrt{2}} \frac{1}{s_{15}s_{34}} T^a_{i\bar{m}} T^b_{m\bar{\jmath}} T^b_{k\bar{l}} \, \langle 1|\varepsilon_5|1+5|3\rangle \, [24] = \frac{c_1 n_1}{D_1}$$



$$= -\frac{i}{\sqrt{2}} \frac{1}{s_{25}s_{34}} T^b_{i\bar{m}} T^a_{m\bar{\jmath}} T^b_{k\bar{l}} \, \langle 13\rangle \, [2|\varepsilon_5|2+5|4] = \frac{c_2 n_2}{D_2}$$



$$= \frac{i}{\sqrt{2}} \frac{1}{s_{12}s_{34}} \tilde{f}^{abc} T^b_{i\bar{\jmath}} T^c_{k\bar{l}} \Big(\langle 1|\,\varepsilon_5|2]\,\langle 3|\,5|4] - \langle 1|\,5|2]\,\langle 3|\,\varepsilon_5|4] - 2\,\langle 13\rangle\,[24]((k_1+k_2)\cdot\varepsilon_5) \Big) = \frac{c_5 n_5}{D_5}$$

Not gauge invariant, but satisfy color-kinematics duality

$$c_1 - c_2 = -c_5 \qquad \Leftrightarrow \qquad n_1 - n_2 = -n_5$$

Color decomposition

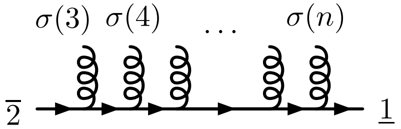
$\mathrm{SU}(N_c)$ basis decomposition

Mangano, Parke, Xu;
Berends, Giele;
Mangano; Kosower + more

only gluons:
$$\mathcal{A}^{\mathrm{tree}}_{n,0} = \sum_{\sigma\in S_{n-1}(\{2,\dots,n\})} \mathrm{Tr}\big(T^{a_1}T^{a_{\sigma(2)}}\dots T^{a_{\sigma(n)}}\big)\, A(1,\sigma(2),\dots,\sigma(n))$$

with quarks more complicated
$$\sim \frac{1}{N_c^p}\big(T^{a_{2k+1}}\dots T^{a_{l_1}}\big)_{i_1\bar{\alpha}_1}\big(T^{a_{l_1+1}}\dots T^{a_{l_2}}\big)_{i_2\bar{\alpha}_2}\cdots\big(T^{a_{l_{k-1}+1}}\dots T^{a_n}\big)_{i_k\bar{\alpha}_k}$$

e.g. *k*=1
$$\mathcal{A}^{\mathrm{tree}}_{n,1} = \sum_{\sigma\in S_{n-2}(\{3,\dots,n\})} \big(T^{a_{\sigma(3)}}\dots T^{a_{\sigma(n)}}\big)_{\bar{\jmath}_2 i_1}\, A(\underline{1},\overline{2},\sigma(3),\dots,\sigma(n))$$



Del Duca, Dixon, Maltoni (DDM) basis

$$\mathcal{A}^{\mathrm{tree}}_{n,0} = \sum_{\sigma\in S_{n-2}(\{3,\dots,n\})} \tilde{f}^{\,a_2 a_{\sigma(3)} b_1}\,\tilde{f}^{\,b_1 a_{\sigma(4)} b_2}\dots\tilde{f}^{\,b_{n-3} a_{\sigma(n)} a_1}\, A(1,2,\sigma(3),\dots,\sigma(n))$$

Properties: valid for any G, gives small *(n - 2)!* basis

Dyck words

Basis of planar (color-ordered) tree amplitudes:

only quarks
$$\left\{ A(\underline{1},\overline{2},\sigma) \;\middle|\; \sigma \in \mathrm{Dyck}_{k-1} \right\}$$
T. Melia

six-point example:

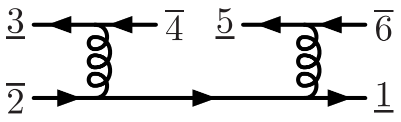
$$\begin{aligned} \mathrm{XYXY} &\;\Rightarrow\; (\underline{3},\overline{4},\underline{5},\overline{6}),\ (\underline{5},\overline{6},\underline{3},\overline{4}) \;\Leftrightarrow\; \{3\,4\}\{5\,6\},\ \{5\,6\}\{3\,4\}\,, \\ \mathrm{XXYY} &\;\Rightarrow\; (\underline{3},\underline{5},\overline{6},\overline{4}),\ (\underline{5},\underline{3},\overline{4},\overline{6}) \;\Leftrightarrow\; \{3\{5\,6\}4\},\ \{5\{3\,4\}6\}\,. \end{aligned}$$

basis: $A(\underline{1},\overline{2},\underline{3},\overline{4},\underline{5},\overline{6})\,,\ A(\underline{1},\overline{2},\underline{5},\overline{6},\underline{3},\overline{4})\,,\ A(\underline{1},\overline{2},\underline{3},\underline{5},\overline{6},\overline{4})$ and $A(\underline{1},\overline{2},\underline{5},\underline{3},\overline{4},\overline{6})$

Color coefficients:

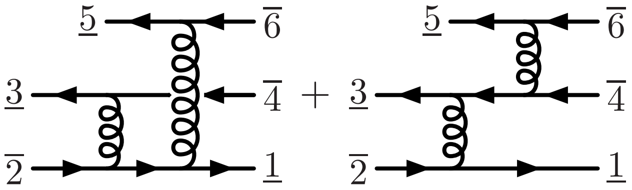
HJ, Ochirov

$$C_{\underline{1}\overline{2}\underline{3}\overline{4}\underline{5}\overline{6}} =$$

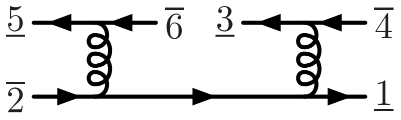


,

$$C_{\underline{1}\overline{2}\underline{3}\underline{5}\overline{6}\overline{4}} =$$

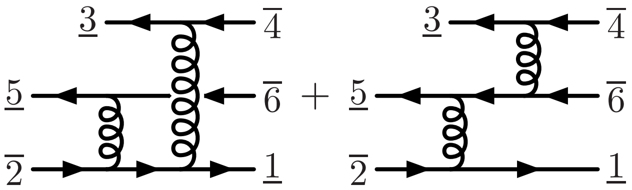


$$C_{\underline{1}\overline{2}\underline{5}\overline{6}\underline{3}\overline{4}} =$$



,

$$C_{\underline{1}\overline{2}\underline{5}\underline{3}\overline{4}\overline{6}} =$$



Melia basis

Basis of planar (color-ordered) tree amplitudes:

gluons & quarks $\left\{ A(\underline{1}, \overline{2}, \sigma) \;\middle|\; \sigma \in \mathrm{Dyck}_{k-1} \times \{\text{gluon insertions}\}_{n-2k} \right\}$

T. Melia

size of basis:

$$\varkappa(n,k) = \underbrace{\overbrace{\frac{(2k-2)!}{k!(k-1)!}}^{\text{empty brackets}} \times (k-1)!}_{\text{dressed quark brackets}} \times \underbrace{(2k-1)(2k)\ldots(n-2)}_{\text{insertions of } (n-2k) \text{ gluons}} = \frac{(n-2)!}{k!}$$

$k \setminus n$	3	4	5	6	7	8
0	1	2	6	24	120	720
1	1	2	6	24	120	720
2	-	1	3	12	60	360
3	-	-	-	4	20	120
4	-	-	-	-	-	30

Color decomposition, any G_c, k, any rep.

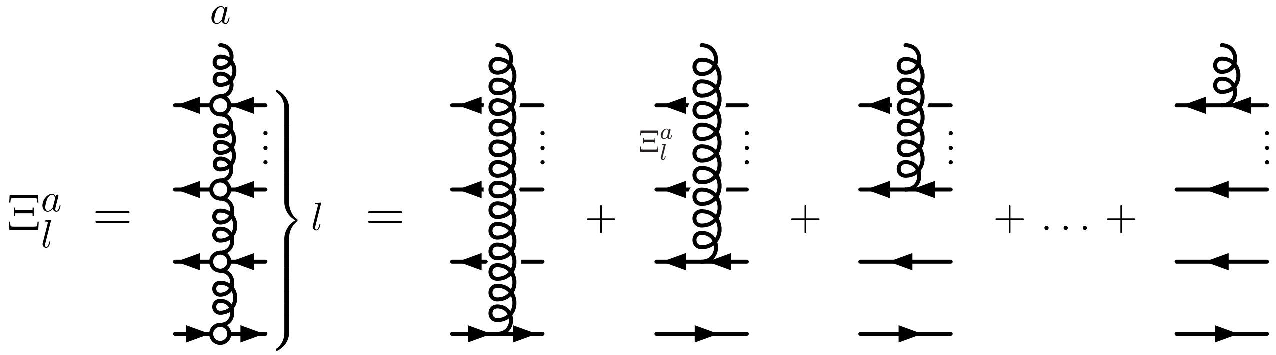
$$\mathcal{A}^{\text{tree}}_{n,k} = \sum_{\sigma \in \text{Melia basis}}^{\varkappa(n,k)} C(\underline{1}, \overline{2}, \sigma)\, A(\underline{1}, \overline{2}, \sigma)$$

HJ, Ochirov

Tensor representations

Tensor l copies of the gauge group Lie algebra:

$$\Xi^a_l = \sum_{s=1}^{l} \underbrace{1 \otimes \cdots \otimes 1 \otimes \overbrace{T^a \otimes 1 \otimes \cdots \otimes 1 \otimes \overline{1}}^{s}}_{l}$$



The Ξ^a_l are Lie algebra generators

$$\left[\Xi^a_l\,,\ \Xi^b_l\right] = \tilde{f}^{abc}\,\Xi^c_l$$

Color coefficients

Color coefficients are given by ‘sandwich’ formulas:

color wave function

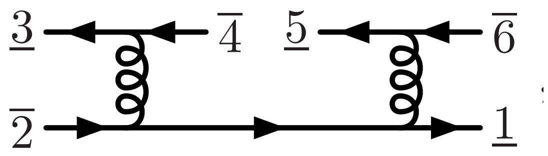
$$C(\underline{1},\overline{2},\sigma)=(-1)^{k-1}\,\{2|\sigma|1\}\Bigg|\begin{array}{lll}\underline{q} & \to & \{q|\,T^b\otimes\Xi^b_{l-1}\\ \overline{q} & \to & |q\}\\ g & \to & \Xi^{a_g}_l\end{array}$$

HJ, Ochirov

(proof by Melia)

For example, consider:

$$C_{\underline{1}\overline{2}\underline{3}\overline{4}\underline{5}\overline{6}}=$$



$$\begin{aligned}C_{\underline{1}\overline{2}\underline{3}\overline{4}\underline{5}\overline{6}}&=\{2|\{3|T^a\otimes\Xi^a_1|4\}\{5|T^b\otimes\Xi^b_1|6\}|1\}=\{2|\{3|T^a\otimes\overline{T}^a|4\}\{5|T^b\otimes\overline{T}^b|6\}|1\}\\&=\{2|\overline{T}^a\overline{T}^b|1\}\{3|T^a|4\}\{5|T^b|6\}=(T^bT^a)_{i_1\bar{\imath}_2}T^a_{i_3\bar{\imath}_4}T^b_{i_5\bar{\imath}_6}\,,\end{aligned}$$

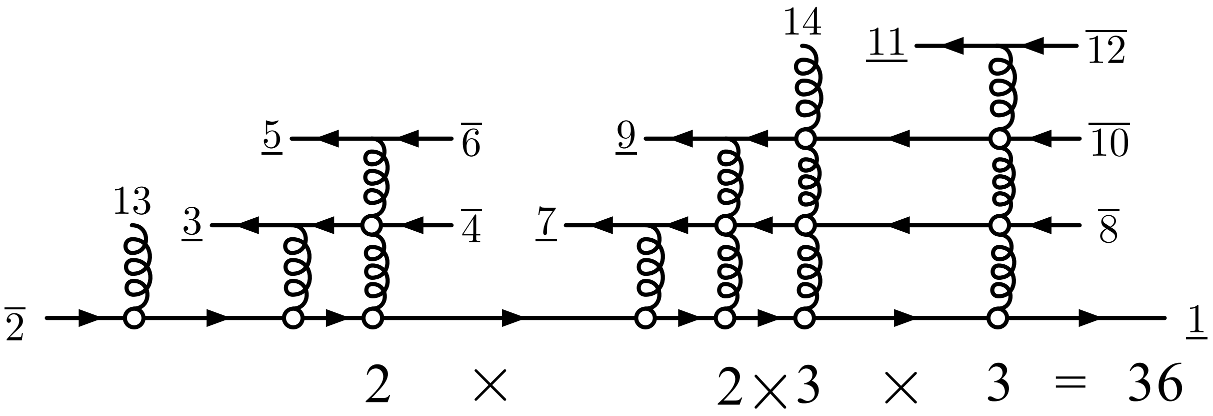
Color coefficient diagrams

Consider a high-multiplicity example:

$$A(\underline{1}, \overline{2}, 13, \underline{3}, \underline{5}, \overline{6}, \overline{4}, \underline{7}, \underline{9}, 14, \underline{11}, \overline{12}, \overline{10}, \overline{8})$$

bra-(c)-ket structure

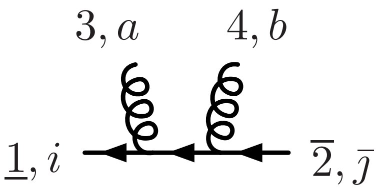
$$\{2\ 13\{3\{5\ 6\}4\}\{7\{9\ 14\{11\ 12\}10\}8\}1\}$$



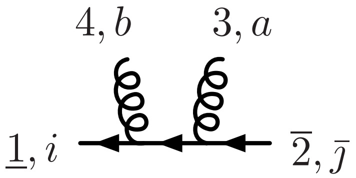
$$C_{\underline{1},\overline{2},13,\underline{3},\underline{5},\overline{6},\overline{4},\underline{7},\underline{9},14,\underline{11},\overline{12},\overline{10},\overline{8}} = -\{2|\Xi_1^{a_{13}}\{3|T^b\otimes \Xi_1^b\{5|T^c\otimes \Xi_2^c|6\}|4\}$$

$$\times\{7|T^d\otimes \Xi_1^d\{9|(T^e\otimes \Xi_2^e)\Xi_3^{a_{14}}\{11|T^f\otimes \Xi_3^f|12\}|10\}|8\}|1\}$$

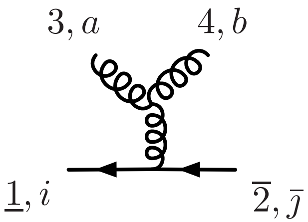
Amplitude relations: example



$$= -\frac{i}{2}\,\frac{T^a_{i\bar k}T^b_{k\bar\jmath}}{s_{13}-m^2}\,(\bar u_1 \not\varepsilon_3 (\not k_{1,3}+m) \not\varepsilon_4 v_2) = \frac{c_1 n_1}{D_1}$$



$$= \frac{c_2 n_2}{D_2}$$



$$= \frac{c_3 n_3}{D_3}$$

commutation rel. holds: $c_1 - c_2 = c_3 \qquad n_1 - n_2 = n_3$

$$\mathcal{A}^{\text{tree}}_{4,1} = \sum_{i=1}^{3} \frac{c_i n_i}{D_i} = \left\{ c_1 \left(\frac{n_1}{D_1} + \frac{n_3}{D_3} \right) + c_2 \left(\frac{n_2}{D_2} - \frac{n_3}{D_3} \right) \right\} \equiv c_2 A_{\underline{1}\overline{2}34} + c_1 A_{\underline{1}\overline{2}43}$$

Gaussian elimination of n_1 $A_{\underline{1}\overline{2}34} = \underbrace{\left(\frac{1}{D_2} + \frac{1}{D_3} - \frac{D_1}{(D_1+D_3)D_3} \right)}_{=0} n_2 - \frac{D_1}{D_1+D_3} A_{\underline{1}\overline{2}43}\,.$

→ **BCJ amplitude rel.** $(s_{14} - m^2) A_{\underline{1}\overline{2}34} = (s_{13} - m^2) A_{\underline{1}\overline{2}43}$

Amplitude relations & basis

BCJ relations for pure-gluon amplitudes:

Bern, Carrasco, HJ

$$\sum_{i=2}^{n-1}\Big(\sum_{j=2}^{i} s_{jn}\Big)A(1,2,\ldots i,n,i+1,\ldots,n-1)=0$$

BCJ relations for quark-gluon QCD amplitudes:

HJ, Ochirov

$$\sum_{i=2}^{n-1}\Big(\sum_{j=2}^{i} s_{jn}-m_j^2\Big)A(1,2,\ldots i,\enclose{circle}{n},i+1,\ldots,n-1)=0$$

gluon !

proof by: de la Cruz,
Kniss,
Weinzierl

Basis:

$k\setminus n$	3	4	5	6	7	8
0	1	1	2	6	24	120
1	1	1	2	6	24	120
2	-	1	2	6	24	120
3	-	-	-	4	16	80
4	-	-	-	-	-	30

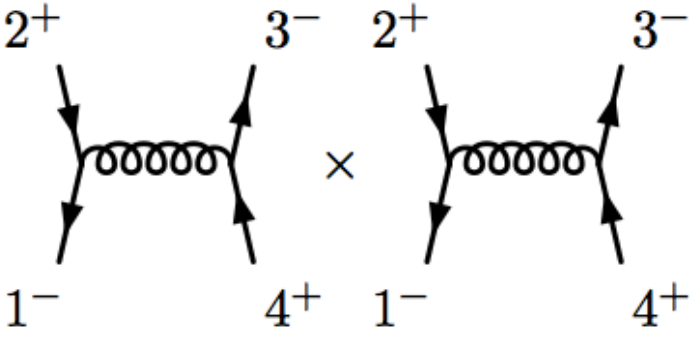
$$(n-3)! \qquad \text{for } k=0,1$$

$$(n-3)!(2k-2)/k! \qquad \text{for } 2<2k\leq n$$

Gravity double copy

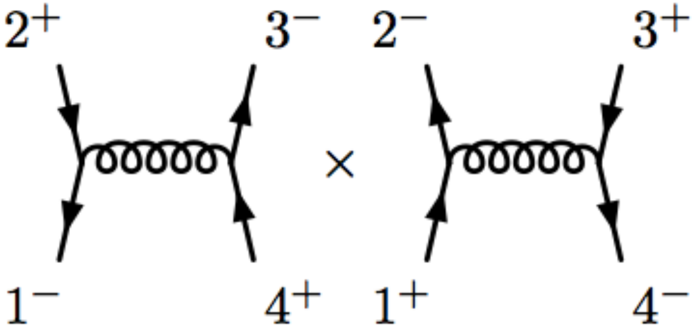
Gravity amplitudes from double copy

Four-photon amplitude in GR (distinguishable matter):



$$A(1^-_\gamma, 2^+_\gamma, 3^-_{\gamma'}, 4^+_{\gamma'}) = \frac{n_s^2}{s} = \frac{\langle 13\rangle^2[24]^2}{s}$$

Four-scalar amplitude in GR (distinguishable matter):



$$A(1^{-+}_\phi, 2^{+-}_\phi, 3^{-+}_{\phi'}, 4^{+-}_{\phi'}) = \frac{n_s\bar{n}_s}{s} = \frac{u^2}{s}$$

indistinguishable matter:

$$A(1^{-+}_\phi, 2^{+-}_\phi, 3^{-+}_{\phi}, 4^{+-}_{\phi}) = \frac{n_s\bar{n}_s}{s} + \frac{n_s\bar{n}_s}{t} = \frac{u^2}{s} + \frac{u^2}{t}$$

Gravity Theories

$$\mathrm{QCD} \otimes \mathrm{QCD} = \mathrm{GR} + \mathrm{matter}$$

matter that only interacts gravitationally

HJ, Ochirov ('14 - '15)

E.g. Maxwell-Einstein theory

Used recently to construct magical and homogeneous *N*=2 sugras

$$\begin{aligned}(\mathcal{N} = 2\ \mathrm{SQCD}) \otimes (D = 7, 8, 10, 14\ \mathrm{QCD}) \\ = \mathrm{Magical}\ \mathcal{N} = 2\ \mathrm{Supergravity} \\ (\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}\ \mathrm{type})\end{aligned}$$

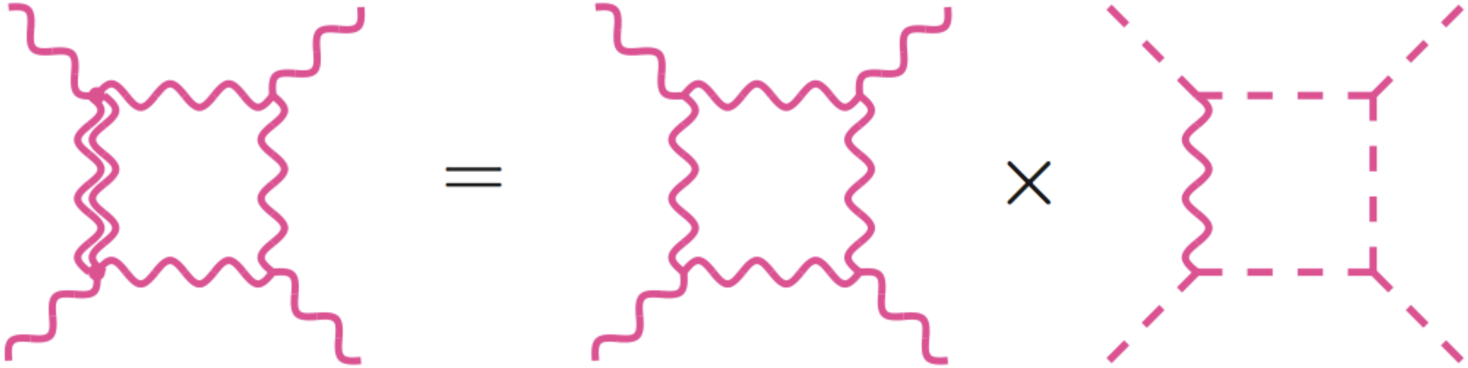
Chiodaroli, Gunaydin, HJ, Roiban ('15)

YM-Einstein theory

Chiodaroli, Gunaydin, HJ, Roiban ('14)

$$\mathrm{GR} + \mathrm{YM} = \mathrm{YM} \otimes (\mathrm{YM} + \phi^3)$$

GR+YM amplitudes are "heterotic" double copies



$\mathcal{N}$= 0,1,2,4 YM-E supergravity

$\mathcal{N}$= 0,1,2,4 SYM

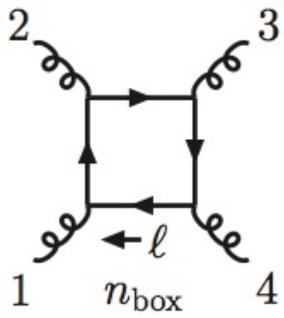
YM + ϕ^3

Gives a construction of the simplest type of gauged supergravities, other constructions should exists for other gaugings.

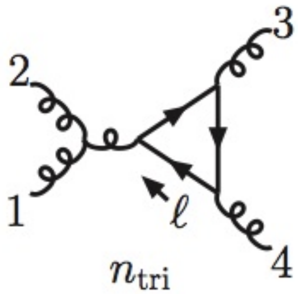
Simple 1-loop examples

One-loop calculations

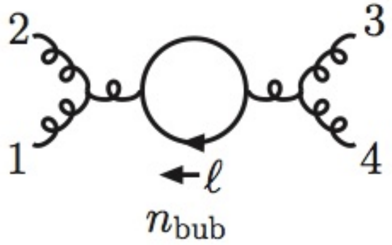
diagrams:



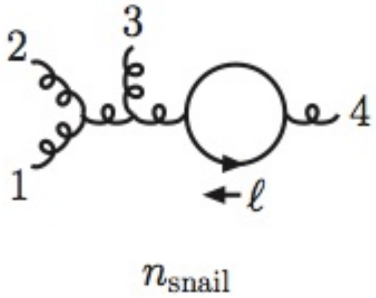
n_{box}



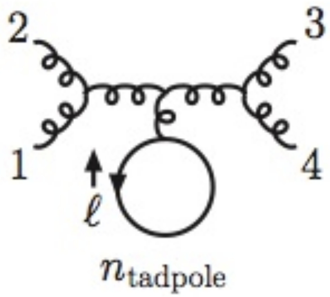
n_{tri}



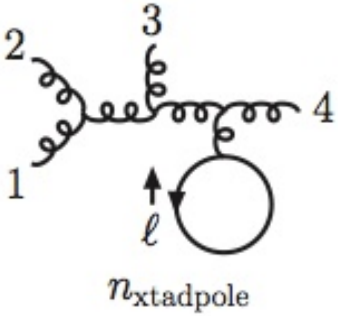
n_{bub}



n_{snail}



n_{tadpole}



n_{xtadpole}

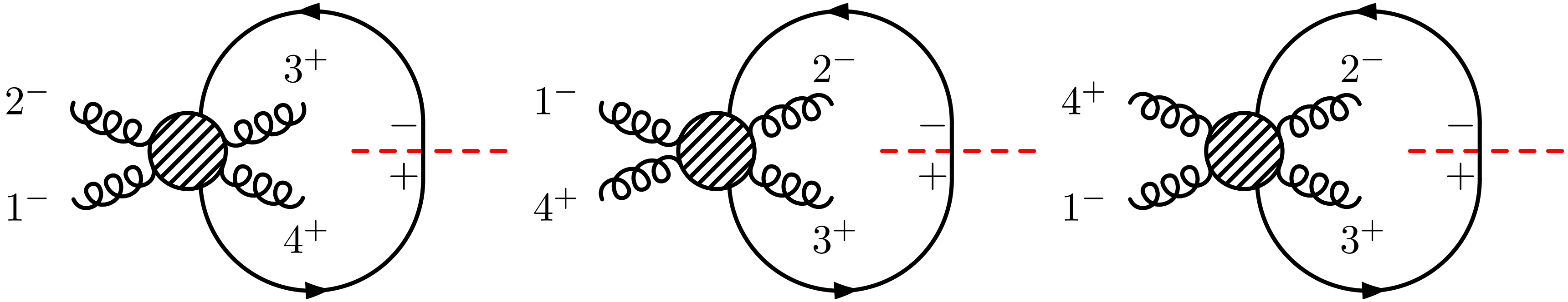
vanish
after
integration

kinematic algebra:

$$
\begin{aligned}
n_{\text{tri}}(1,2,3,4,\ell) &= n_{\text{box}}([1,2],3,4,\ell)\,,\\
n_{\text{bub}}(1,2,3,4,\ell) &= n_{\text{box}}([1,2],[3,4],\ell)\,,\\
n_{\text{snail}}(1,2,3,4,\ell) &= n_{\text{box}}([[1,2],3],4,\ell)\,,\\
n_{\text{tadpole}}(1,2,3,4,\ell) &= n_{\text{box}}([[1,2],[3,4]],\ell)\,,\\
n_{\text{xtadpole}}(1,2,3,4,\ell) &= n_{\text{box}}([[[1,2],3],4],\ell)\,.
\end{aligned}
$$

Unitarity cuts

Parameters in ansatz fixed by unitarity cuts (unitarity method) Bern, Dixon, Dunbar, Kosower



***N*=2 SQCD:** Carrasco, Chiodaroli, Gunaydin, Roiban; Nohle; Ochirov, Tourkine, HJ, Ochirov

$$n_{\text{box}}^{\mathcal{N}=2,\text{fund}} = (\kappa_{12}+\kappa_{34})\frac{(s-\ell_s)^2}{2s^2} + (\kappa_{23}+\kappa_{14})\frac{\ell_t^2}{2t^2} + (\kappa_{13}+\kappa_{24})\frac{st+(s+\ell_u)^2}{2u^2}$$
$$- 2i\epsilon(1,2,3,\ell)\frac{\kappa_{13}-\kappa_{24}}{u^2} + \mu^2\left(\frac{\kappa_{12}+\kappa_{34}}{s} + \frac{\kappa_{23}+\kappa_{14}}{t} + \frac{\kappa_{13}+\kappa_{24}}{u}\right)$$

***N*=1 SQCD:** HJ, Ochirov

$$n_{\text{box}}^{\mathcal{N}=1,\text{odd}} = (\kappa_{12}-\kappa_{34})\frac{(\ell_s-s)^3}{2s^3} + (\kappa_{23}-\kappa_{14})\frac{\ell_t^3}{2t^3} + (\kappa_{13}-\kappa_{24})\frac{1}{2}\left(\frac{\ell_u^3}{u^3} + \frac{3s\ell_u^2}{u^3} - \frac{3s\ell_u}{u^2} + \frac{s}{u}\right)$$
$$- 2i\epsilon(1,2,3,\ell)(\kappa_{13}+\kappa_{24})\frac{2\ell_u-u}{u^3} - a\mu^2(\kappa_{13}-\kappa_{24})\frac{s-t}{u^2}\,,$$

YM + scalar: Nohle; HJ, Ochirov

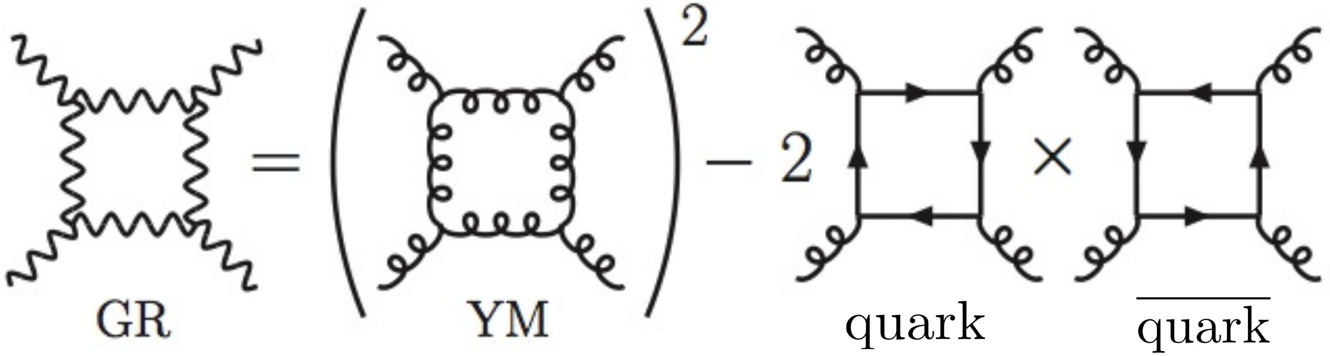
QCD: HJ, Ochirov

Using the QCD numerators to get GR

Pure Einstein gravity can be obtained from the QCD numerators:

$$\mathcal{M}_4^{(1)} = \sum_{\mathcal{S}_4} \sum_{i=\{B,t,b\}} \int \frac{d^D\ell}{(2\pi)^D} \frac{1}{S_i} \frac{n_i^V n_i{}^{V\prime} - \overline{n}_i^m n_i{}^{m\prime} - n_i^m \overline{n}_i{}^{m\prime}}{D_i}$$

The YM square contains dilaton & axion, which has to be subtracted out



HJ, Ochirov

...and similarly for triangle and bubble

Gives correct pure GR amplitude (cf. Dunbar & Norridge)

Summary

- Color-kinematics duality implies kinematic Lie algebra relations satisfied by the numerators of gauge theory amplitudes
- Generalized color-kinematics duality to QCD tree amplitudes
- New color decomposition of QCD tree amplitudes
- BCJ amplitude relations between primitives of QCD
- Checks: Explicitly up to 8pts tree level, proof color decompositon (Melia)
  proof BCJ relations (Weinzierl, et al.)
- Constructed one-loop 4pt amplitude in *N*=1,2 SQCD and QCD such that the duality is manifest.
- Useful for construction of QCD loop amplitudes as well as for pure Einstein and gravity + matter amplitudes